COMPLETELY CONTINUOUS HANKEL OPERATORS ON H^∞ AND BOURGAIN ALGEBRAS

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ABSTRACT. Let $(H^\infty)_b$ be the Bourgain algebra of $H^\infty \subset L^\infty$. We prove $(H^\infty)_b = H^\infty + C$. In particular if $f \in L^\infty$ then the Hankel operator H_f is a compact map of H^∞ into BMO iff whenever $f_n \rightarrow 0$ weakly in H^∞ , then $\text{dist}(ff_n, H^\infty) \rightarrow 0$.

1. INTRODUCTION AND RESULTS

Cima and Timoney [3] introduced the following definition, based on an idea of Bourgain [1]: Let A be a Banach algebra and let X be a linear subspace of A . Then the "Bourgain algebra" X_b is defined to be the set of all $x \in A$ such that

$$(1.1) \quad \text{if } x_n \rightarrow 0 \text{ weakly in } X, \text{ then } \text{dist}(xx_n, X) \rightarrow 0.$$

It is proved in Cima and Timoney [3] that X_b is a closed subalgebra of A , and that if X is an algebra then $X_b \supset X$. (Actually, Cima and Timoney only consider the case $A = C(K)$, but the same proof holds in general.) Bourgain [2] has shown that $H^\infty(D)$ has the Dunford-Pettis property using the theory of ultra-products. A Banach space X has the Dunford-Pettis property if whenever $x_n \rightarrow 0$ weakly in X and $x_n^* \rightarrow 0$ weakly in X^* then

$$\lim_{n \rightarrow \infty} \langle x_n, x_n^* \rangle = 0.$$

In trying to develop a more basic function theoretic proof of this result the following question arose. If $A = L^\infty(T)$ and $X = H^\infty$, characterize the closed subalgebra X_b , $H^\infty \subset X_b \subset L^\infty$; equality $X_b = L^\infty$ implying that H^∞ has the Dunford-Pettis property. One of our main results is that then $X_b = H^\infty + C$.

We will show this by translating (1.1) into a condition for Hankel operators. We review first some pertinent definitions.

P denotes the orthogonal projection of $L^2(T)$ onto the subspace $H^2(T)$ of analytic functions.

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If $f \in L^\infty$, then the Hankel operator H_f is defined by

$$H_f(g) = (I - P)(fg), \quad g \in L^2.$$

For the definition of BMO and its subspace VMO we refer to Garnett [4, Chapter VI]. We note that BMO is a linear space (containing L^∞) and that it follows that [4, Theorem VI.1.5] that $P, I - P: L^\infty \rightarrow \text{BMO}$. Hence, for $f \in L^\infty$, H_f maps L^∞ into BMO.

We finally recall that a completely continuous operator is a linear operator which maps weakly convergent sequences into strongly convergent ones. (This is Hilbert's definition; for reflexive spaces it is equivalent to compactness, see §2.)

We may now summarize our results as follows.

Theorem 1. *Let $f \in L^\infty$. Then the following are equivalent.*

- (i) *if $f_n \rightarrow 0$ weakly in H^∞ , then $\text{dist}(ff_n, H^\infty) \rightarrow 0$.*
- (ii) *H_f is a completely continuous map of H^∞ into BMO.*
- (iii) *H_f is a compact map of H^∞ into BMO.*
- (iv) *$f \in H^\infty + C$.*
- (v) *$(I - P)f \in \text{VMO}$.*

Here it is important that we deal with sequences in the conditions (i) and (ii), and not with general nets. In fact, we have the following companion result, where R denotes the set of all rational functions with no poles on T .

Theorem 2. *Let $f \in L^\infty$. Then the following are equivalent.*

- (i) *if $\{f_\alpha\}$ is any net which converges weakly to 0 in H^∞ , then $\text{dist}(ff_\alpha, H^\infty) \rightarrow 0$.*
- (ii) *H_f is a continuous map of (H^∞, weak) into $(\text{BMO}, \text{norm})$.*
- (iii) *H_f maps H^∞ into a finite dimensional subspace of BMO.*
- (iv) *$(I - P)f \in R$.*

Remark 1. The theorems remain valid if we replace both H^∞ and BMO by H^2 throughout the statements. The equivalences (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) in Theorem 1 then is Hartman's theorem, and the equivalences (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) in Theorem 2 is Kronecker's theorem. The proofs simplify in this case, in particular (ii) $\Leftrightarrow$ (iii) in Theorem 1 because H^2 is reflexive.

The same results hold for any H^p , $1 < p < \infty$.

Remark 2. The extensions in Remark 1 can be regarded as examples of a generalization of the Bourgain algebra as defined above, where instead of assuming $X \subset A$, we assume that X is a subspace of an A -module M . (Let $A = L^\infty$, $M = L^p$ and $X = H^p$.) The proof in Cima and Timoney [3] shows that (1.1) defines a closed subalgebra of A also in this case.

Remark 3. We obtain the same results with weak * -convergence in (i) and (ii) in the theorems. (This makes the conditions stronger, but it is easy to see that they still follow from (iv).)

2. PROOF OF THEOREM 1

We divide the proof into several steps.

(i) $\Leftrightarrow$ (ii). It is an easy consequence of Fefferman's duality theorem (cf. Garnett [4, Corollary VI.4.6]) that

$$(2.1) \quad \text{dist}(ff_n, H^\infty) \sim \|(I - P)(ff_n)\|_{\text{BMO}} = \|H_f(f_n)\|_{\text{BMO}}.$$

(iii) $\Rightarrow$ (ii). This is a consequence of the following elementary lemma.

Lemma 1. *Let X and Y be Banach spaces. A linear mapping T is completely continuous if and only if it maps (relatively) weakly compact sets to (relatively) compact sets.*

Proof. The proof is elementary and probably known.

In particular, any compact map is completely continuous (which is what we need here), and the converse holds if X is reflexive (see e.g. Rudin [5, Exercise IV.18]).

(iv) $\Rightarrow$ (iii). If $f = h + p$, where $h \in H^\infty$ and p is polynomial in $\bar{z}$, then H_f has finite rank. The implication follows because such functions are dense in $H^\infty + C$, and the mapping $f \rightarrow H_f$ is a continuous map of L^∞ into $L(H^\infty, \text{BMO})$.

(iv) $\Leftrightarrow$ (v). This is well known, and another consequence of Fefferman's duality theorem; cf. Garnett (1981, §VI.5).

(i) $\Rightarrow$ (iv). By the results of Cima and Timoney [3] quoted in the introduction, the set $(H^\infty)_b$ of functions satisfying (i) is a closed subalgebra of L^∞ containing H^∞ . By the Chang-Marshall theorem (Garnett [4, §IV.3]), every such algebra is generated by H^∞ and those complex conjugates of interpolating Blaschke products which belong to the algebra. Therefore, it suffices to show that $(H^\infty)_b$ does not contain any conjugate of an infinite interpolating Blaschke product.

In order to do this, let $\{z_n\}$ be an interpolating sequence in the unit disc and let b be the corresponding Blaschke product. By the P. Beurling interpolation theorem (Garnett [4, Theorem VII.2.1]), there exist functions $f_n \in H^\infty$ such that $f_n(z_m) = \delta_{nm}$ and

$$(2.2) \quad \sum_n |f_n(z)| \leq M, \quad |z| < 1.$$

We first observe that (2.2) implies that $f_n \rightarrow 0$ weakly in H^∞ . In fact, if $\varphi \in (H^\infty)^*$, let $\alpha_n = \text{sign} \langle \varphi, f_n \rangle$. Then, for any $N < \infty$,

$$\sum_1^N |\langle \varphi, f_n \rangle| = \sum_1^N \alpha_n \langle \varphi, f_n \rangle = \left\langle \varphi, \sum_1^N \alpha_n f_n \right\rangle \leq \|\varphi\| \left\| \sum_1^N \alpha_n f_n \right\| \leq \|\varphi\| \cdot M.$$

Consequently $\sum_1^\infty |\langle \varphi, f_n \rangle| < \infty$, and thus $\langle \varphi, f_n \rangle \rightarrow 0$, for any $\varphi \in (H^\infty)^*$.

Next we observe that f_n/b is analytic in the unit disc except at z_n , where it has a simple pole with residue $1/b'(z_n)$. Hence $f_n/b - 1/b'(z_n)(z - z_n) \in H^\infty$ and, since $\bar{b}f_n = f_n/b$ on T ,

$$(2.3) \quad \begin{aligned} \text{dist}(\bar{b}f_n, H^\infty) &= |b'(z_n)|^{-1} \text{dist}(1/(z - z_n), H^\infty) \\ &\sim |b'(z_n)|^{-1} \|(I - P)(1/(z - z_n))\|_{\text{BMO}}. \end{aligned}$$

Furthermore,

$$(2.4) \quad |b'(z_n)| = \frac{1}{1 - |z_n|^2} \prod_{m \neq n} \left| \frac{z_n - z_m}{1 - \bar{z}_n \bar{z}_m} \right| \leq (1 - |z_n|^2)^{-1}$$

and (for $z \in T$)

$$\frac{1}{z - z_n} = \frac{\bar{z}}{1 - z_n \bar{z}} \in \overline{(H_0^\infty)}$$

so that $P(1/(z - z_n)) = 0$ and

$$\begin{aligned} (1 - |z_n|^2) \left\| (I - P) \frac{1}{z - z_n} \right\|_{\text{BMO}} &= (1 - |z_n|^2) \left\| \frac{\bar{z}}{1 - z_n \bar{z}} \right\|_{\text{BMO}} \\ &= \left\| \frac{\bar{z} - \bar{z} z_n \bar{z}_n}{1 - z_n \bar{z}} \right\|_{\text{BMO}} \\ &= \left\| \frac{(\bar{z} - \bar{z}_n) + \bar{z}_n (1 - z_n \bar{z})}{1 - z_n \bar{z}} \right\|_{\text{BMO}} \\ &= \left\| \frac{\bar{z} - \bar{z}_n}{1 - z_n \bar{z}} + \bar{z}_n \right\|_{\text{BMO}} \\ &= \left\| \frac{\bar{z} - \bar{z}_n}{1 - z_n \bar{z}} \right\|_{\text{BMO}} \\ &= \|\bar{z}\|_{\text{BMO}}, \end{aligned}$$

independently of n (where we in the last step used the fact that a suitably chosen norm on BMO is Möbius invariant (Garnett [4, Corollary VI.1.4]).

(2.3)–(2.5) now show that $\text{dist}(\bar{b}f_n, H^\infty) \geq C > 0$ for all n , and thus $\bar{b} \notin (H^\infty)_b$. This completes the proof of Theorem 1.

3. PROOF OF THEOREM 2

(i) $\Leftrightarrow$ (ii). By (2.1).

(i) $\Leftrightarrow$ (iii). We formulate this as a general result.

Lemma 2. *If X and Y are normed spaces and T is a bounded linear map of X into Y , then T is continuous $(X, \text{weak}) \rightarrow (Y, \text{norm})$ if and only if T has finite rank.*

Proof. An elementary exercise.

(iii) $\Leftrightarrow$ (v). Since H^∞ is a dense subspace of H^2 , (iii) holds if and only if H_f has finite rank on H^2 , which is equivalent to (v) by Kronecker's theorem.

(iv) $\Leftrightarrow$ (v). Well known and elementary.

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