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BANACH LATTICES WITH THE SUBSEQUENCE SPLITTING PROPERTY

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ABSTRACT. A Banach lattice X has SSP if every bounded sequence in X has a subsequence that splits into a X-equi-integrable sequence and a sequence with pairwise disjoint support. We characterize such lattices in terms of uniform order continuity conditions and ultrapowers. This implies that rearrangement invariant function spaces with the Fatou-property have SSP.

0. INTRODUCTION

In [11] Kadec and A. Pełczyński made the observation that every bounded sequence (f_n) of $L_p[0,1]$, $1 \leq p < \infty$, has a subsequence that splits into two 'extreme' sequences (g_k) and (h_k), where the h_k's have pairwise disjoint support and the g_k's are L_p-equi-integrable, i.e. $\sup_k \|\chi_A g_k\|_{L_p} \to 0$ for $\mu(A) \to 0$.

They used this fact to study the subspace structure of $L_1[0,1]$. Soon, this idea proved to be useful also in various other contexts and it was observed that the above splitting is possible in more general Banach function spaces. E.g. in [4 and 10] such splittings appear in the study of isomorphic embeddings of Banach function spaces and it was pointed out that they are possible in Orlicz function spaces with the Δ_2-condition and in q-concave lattices (cf. [7]). Equi-integrable and pairwise disjoint sequences play an important part in the study of compactness properties of positive operators (see e.g. [6, 7, 15 and 13]) and, more recently, such splittings of sequences were considered in the context of ergodic theorems for positive contractions in [1]. It follows from this work, that the above splitting is also possible in Banach lattices with uniformly monotone norm.

On the other hand, there are Banach lattices with sequences for which the splitting is not possible. A simple example is c_0, but Figiel, Ghoussoub and Johnson also have constructed reflexive, p-convex Banach lattices without the subsequence splitting property. This raises the question: what kind of structural

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property of Banach lattices makes the splitting work? In this paper we give two characterizations of the subsequence splitting property. One of them in terms of a 'uniform' order continuity of X (Theorem 2.8), the second in terms of ultrapowers $X_{\mathfrak{A}}$ of X (Theorem 2.5). While the sufficient conditions we quoted in the first paragraph require the whole ultrapower $X_{\mathfrak{A}}$ to have order continuous norm, we show that order-continuity of a relatively 'small' band $\widetilde{X}$ of $X_{\mathfrak{A}}$ is already necessary and sufficient. In a sense made precise in §1, $\widetilde{X}$ is the 'same' function space as X on a 'larger' measure space.

We also find a large new class of spaces with the subsequence splitting property, namely all rearrangement invariant function spaces with the Fatou-property (equivalently, with no subspace isomorphic to c_0, see Corollary 2.6).

1. Preliminaries on ultraproducts and equi-integrable sets

In this paper X is always a Banach lattice with order continuous norm, which has a weak unit. By a general representation theorem (see e.g. [12, 1.6.14]) X can be represented as a lattice ideal of $L_1(\Omega, \Sigma, \mu)$ on some probability space (Ω, Σ, μ) such that

$$L_\infty(\Omega, \Sigma, \mu) \subset X \subset L_1(\Omega, \Sigma, \mu). \tag{1}$$

In the following we always assume that X is given in this way. Let $\mathfrak{A}$ be a free ultrafilter on $\mathbf{N}$. The ultraproduct $X_{\mathfrak{A}}$ of X is defined as the quotient

$$X_{\mathfrak{A}} = l_\infty(X)/M, \qquad M = \{(f_n) \in l_\infty(X) \colon \mathfrak{A}\text{-}\lim \|f_n\| = 0\}.$$

$f = [f_n]$ denotes the equivalence class of the sequence $(f_n) \subset X$ and $\|f\| = \mathfrak{A}\text{-}\lim \|f_n\|_X$. $X_{\mathfrak{A}}$ is also a Banach lattice with $[f_n] \wedge [g_n] = [f_n \wedge g_n]$. (See [9] for details of this construction.)

1.1. **Definition.** By $\widetilde{X}$ we denote the band $\{1\}^{\perp\perp}$ in X generated by $[f_n]$, $f_n \equiv 1$, and $\widetilde{X}^{\perp}$ is the orthogonal band of $\widetilde{X}$ in $X_{\mathfrak{A}}$.

Following a construction in [3] we can represent $\widetilde{X}$ as a Banach function lattice: For $L = L_1(\Omega, \Sigma, \mu)$, $\widetilde{L}$ is an abstract L-space which can be represented as $L_1(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mu})$ in such a way that the elements $[\chi_{A_n}] \in \widetilde{L}$, $A_n \in \Sigma$, correspond to characteristic functions χ_A, $A \in \widetilde{\Sigma}$, with $\widetilde{\mu}(A) = \|[\chi_{A_n}]\|_L = \mathfrak{A}\text{-}\lim \mu(A_n)$. Now the lattice embeddings in (1) extend to lattice embeddings

$$L_\infty(\widetilde{X}, \widetilde{\Sigma}, \widetilde{\mu}) \subset \widetilde{X} \subset L_1(\widetilde{\Omega}, \widetilde{\Sigma}, \widetilde{\mu}). \tag{2}$$

1.2. **Examples.** (a) If X is a purely atomic lattice then $\widetilde{X} = X$.

(b) A rearrangement invariant function space X on (Ω, Σ, μ) is of the form

$$X = \{f \in L_1(\Omega, \mu) \colon f^* \in X_0\}, \qquad \|f\|_X = \|f^*\|_{X_0}$$

where f^* is the decreasing rearrangement of f on $[0, 1]$ and X_0 is a rearrangement invariant function space on $[0, 1]$ (see [2, 18.2]). Then

$$\widetilde{X} = \{f \in L_1(\widetilde{\Omega}, \widetilde{\mu}) \colon f^* \in X_0\}, \qquad \|f\|_{\widetilde{X}} = \|f^*\|_{X_0}.$$

In particular, for $1 \leq q < \infty$: $\widetilde{L_q(\Omega, \mu)} = L_q(\widetilde{\Omega}, \widetilde{\mu})$.

Proof. (a) Choose $\Omega_m \subset \Omega$ such that $\Omega \backslash \Omega_m$ consists of finitely many atoms and $\mu(\Omega_m) \to 0$. If $0 \leq [f_n] \in \widetilde{X}$ then $\beta_m = \mathfrak{A}\text{-}\lim_n \|f_n|_{\Omega_m}\| \to 0$ for $m \to \infty$. Otherwise there are $\varepsilon > 0$ and $U_m \in \mathfrak{A}$ with $U_m \supset U_{m+1} \supset \cdots$, $\bigcap U_m = \varnothing$, such that $\|f_n|_{\Omega_m}\| \geq \varepsilon$ for $n \in U_m$. Then $g = [g_n]$ with $g_n = f_n|_{\Omega_m}$ for $n \in U_m \backslash U_{m+1}$ satisfies $0 \leq g \leq f$ and $g \in \widehat{X}^{\perp}$ which contradicts $\|g\| \geq \varepsilon$. $\beta_m \to 0$ implies that $f = \mathfrak{A}\text{-}\lim f_n$ exists in the norm of X and $[f_n] = [f]$.

(b) Let $0 \leq f \in \widetilde{X}$. By Proposition 1.5 below there is a sequence $f_n \in X$ with $f = [f_n]$ which is equi-integrable in $L_1(\Omega, \mu)$. Denote by $D(t)$ and $D_n(t)$ the distribution functions of $f \in \widetilde{X}$ and $f_n \in X$, i.e. $D(t) = \widetilde{\mu}(|f| > t)$, $D_n(f) = \mu(|f_n| > t)$. The functions D_n, $n \in \mathbf{N}$, are equi-continuous on $[0, \infty)$ and $\mathfrak{A}\text{-}\lim_n D_n(t) = D(t)$ for all $t \in [0, \infty)$ by [3, Proposition 1.3]. Hence $D = \mathfrak{A}\text{-}\lim D_n$ in the sup-norm and

$$\|f\|_{\widetilde{X}} = \mathfrak{A}\text{-}\lim_n \|f_n\|_X = \mathfrak{A}\text{-}\lim \|f_n^*\|_{X_0} = \|f^*\|_{X_0}. \quad \square$$

1.3. **Proposition.** *For all* $f \in \widetilde{X}$ *we have* $\|f\| = \sup_m \|f \wedge m1\|$.

Proof. Let $f = [f_n] \in \widetilde{X}$ with $\|f_n\| = 1$. For $\varepsilon > 0$ choose c_n with $\|f_n \wedge c_n 1\| = 1 - \varepsilon$. We have to show that there is an m with $U_m = \{n : c_n \leq m\} \in \mathfrak{A}$ because then

$$\|f_n \wedge m1\| \geq \|f_n \wedge c_n 1\| = 1 - \varepsilon \quad \text{for all } n \in U_m$$

and the claim follows. Otherwise we have $\mathfrak{A}\text{-}\lim c_n = \infty$ and $g_n = f_n - f_n \wedge c_n 1 \to 0$ in measure. Then $g = [g_n] \in \widetilde{X}^{\perp}$ but also $g \leq f \in \widetilde{X}$. This implies $\mathfrak{A}\text{-}\lim \|g_n\| = 0$ which contradicts $\|g_n\| \geq \varepsilon$. $\square$

1.4. *Remark.* A subset $M \subset X$ is called *equi-integrable in* X if

$$\sup\{\|\chi_A f\| : f \in M, \mu(A) \leq \delta\} \to 0 \quad \text{for } \delta \to 0.$$

It follows from the order continuity of X that M is equi-integrable in X iff $\sup\{\||f| - |f| \wedge t1\| : f \in M\} \to 0$ for $t \to \infty$.

1.5. **Proposition.** *Let* $f = [f_n] \in X_{\mathfrak{A}}$.

(a) f *belongs to* $\widetilde{X}^{\perp}$ *if and only if* $\mathfrak{A}\text{-}\lim f_n = 0$ *with respect to the topology of convergence in measure,*

(b) f *is an order continuous element of* X (*i.e.* $\|\chi_A f\|_{\widetilde{X}} \to 0$ *for* $\widetilde{\mu}(A) \to 0$) *if and only if* f *has a representation* $f = [f_n]$ *with an equi-integrable sequence in* X.

Proof. (a) Since $\widetilde{X}^{\perp} = [1]^{\perp}$ we have that $[f_n] \in \widetilde{X}^{\perp}$ iff

$$\mathfrak{A}\text{-}\lim_n \| |f_n| \wedge m1\| = 0 \quad \text{for all } m.$$

(b) If the sequence (f_n) is equi-integrable in X then

$$\|\chi_A f\|_{\widetilde{X}} \leq \sup_n \{\|\chi_B f_n\| : \mu(B) \leq \widetilde{\mu}(A)\} \to 0 \quad \text{for } \widetilde{\mu}(A) \to 0.$$

Now let $0 \leq f = [f_n] \in \widetilde{X}$ be an order continuous element. Choose a sequence (n_m) such that

$$\text{(3)} \qquad \|f - f \wedge n_m 1\|_{\widetilde{X}} \leq 2^{-m}.$$

Put $g^1 = f \wedge n_1 1$ and choose $g_n^1 \leq n_1 1$ with $g^1 = [g_n^1]$. For each $m > 1$ choose a sequence $(g_n^m)_{n \in \mathbf{N}}$ with

$$\text{(4)} \qquad g^m = [g_n^m] = f \wedge n_{m+1} 1 - f \wedge n_m 1, \qquad g_n^m \leq (n_{m+1} - n_m) 1$$
$$\|g_n^m\|_X = \|g^m\|_{\widetilde{X}} \leq 2^{-m} \quad \text{for all } n.$$

Put $h_n = \sum_m g_n^m \in X$. Since $f \wedge n_m 1 = \sum_{k=1}^{m-1} g^k$ it follows from (3) and (4) that $f = [h_n]$. (h_n) is equi-integrable in X because for all n

$$\sum_{k=1}^{m} g_n^k \leq n_m 1, \qquad \left\| \sum_{k=m+1}^{\infty} g_n^k \right\|_X \leq 2^{-m}. \quad \square$$

1.6. **Notation.** In analyzing equi-integrable sets we need the following 'norm-distribution' of $f \in X$:

$$d_f(t) = \|f\| - \| |f| \wedge t1 \| \quad \text{for } t \in \mathbf{R}^+.$$

Since X is order continuous, d_f is continuous, decreasing and $\lim_{t \to \infty} d_f(t) = 0$. It follows from 1.3 and 1.5(b) that d_f for $f \in \widetilde{X}$ has these properties too.

1.7. **Proposition.** *Let $M \subset X$ be bounded.*

(a) *If M is an equi-integrable set in X, then $\{d_f, f \in M\}$ is a compact subset of $C[0, \infty)$ with respect to uniform convergence.*

(b) *If $f_n \in M$ and $f_n \to 0$ in measure, then*

$$d_{f_n}(t) \to 0 \textit{ for } n \to \infty \quad \textit{for all } t > 0.$$

Proof. (a) Since $|d_f(t) - d_f(s)| \leq \|f\| \cdot |s - t|$ the set $\{d_f, f \in M\}$ is equi-continuous on $C[0, s]$ for all s. Equi-integrability implies that $d_f(t) \to 0$ for $t \to \infty$, uniformly for $f \in M$. (b) clear. $\square$

2. The characterization theorem

Throughout this section, X denotes a Banach lattice with order continuous norm, represented as a function spaces as in (1).

2.1. **Definition.** X has the *subsequence splitting property* (SSP) if for every bounded sequence $(f_n) \subset X$ there is a subsequence (n_k) and sequences (g_k), (h_k) with $|g_k| \wedge |h_k| = 0$ and $f_{n_k} = g_k + h_k$ such that

(i) (g_k) is equi-integrable in X, i.e. $\| |g_k| - |g_k| \wedge t1 \| \to 0$ for $t \to \infty$ uniformly in k.

(ii) The h_k's are pairwise disjoint.

The following examples appear in the literature:

2.2. **Examples.** (a) $L_p(\Omega, \mu)$, $1 \le p < \infty$, and more generally Orlicz-function spaces satisfying the D_2-condition have the SSP. (See [11, 4].)

(b) [7] q-concave Banach lattices with $q < \infty$ have SSP. Recall that X is q-concave if there is a constant C such that for all $f_1, \dots, f_n \in X$

$$\left\|\sum f_n\right\| \ge c\left(\sum \|f_n\|^q\right)^{1/q}.$$

(c) [1] Banach lattices with a uniformly monotone norm have SSP. The norm is uniformly monotone, if for ever $\alpha > 0$ there is a $\beta > 0$ such that for $f, g \in X_+$ with $\|f\| \ge \alpha$ and $\|g\| \le 1$ we have $\|f+g\| \ge \|g\| + \beta$.

(d) Purely atomic Banach lattices have SSP. In this case, SSP is just a restatement of the Bessarga-Pełczyński basis selection principle [5, p. 42].

Since each of the assumptions in (a) to (d) implies that $X_{\mathfrak{A}}$ has order continuous norm, it follows from Theorem 4 below that we have SSP in each of these cases, but we also see that the conditions (a) to (d) are far from being necessary. Indeed, only the relatively small band $\widetilde{X}$ of $X_{\mathfrak{A}}$ has to have order continuous norm.

On the other hand, the following counterexamples are known.

(e) $X = c_0$ does not have SSP. (Just consider $f_n = \sum_{m=1}^n e_m$, $e_m = (\delta_{m,j})_j$.)

(f) [7] There are reflexive, p-convex Banach lattices such that X does not have SSP but X^* has SSP. □

In the following characterization theorem, we need a variant of finite representability (compare [9]).

2.4. **Definition.** We say that l_∞^n's are equi-normably embedded into X if for every $\varepsilon > 0$ there are $f_i^n \in X_+$, $i = 1, \dots, n$, $n \in \mathbf{N}$ with

(i) $\|f_i^n\| = 1$, $f_i^n \wedge f_j^n = 0$ for $i \ne j$,
(ii) $\|\sum_{i=1}^n f_i\| \le 1 + \varepsilon$, i.e. $[f_1^n, \dots, f_n^n]$ is $(1+\varepsilon)$ isomorphic to l_n^∞,
(iii) for every i, the sequence $(d_{f_i^n})_n$ norm converges in $C[0,\infty)$.

2.5. **Theorem.** *For a Banach lattice X the following statements are equivalent:*

(1) *X has the subsequence splitting property.*
(2) *$\widetilde{X}$ has order continuous norm (see Definition 1.1).*
(3) *c_0 does not embed in $\widetilde{X}$.*
(4) *l_∞^n cannot be embedded equi-normably in X.*

Proof. We can assume that c_0 does not imbed into X since this is implied by all of the above conditions.

(1) ⇒ (2). Otherwise, there are pairwise disjoint $A^m \in \widetilde{\Sigma}$, $f = [f_n] \in \widetilde{X}_+$ and $\delta > 0$ such that $\|f\chi_{A^m}\| > \delta$. Choose a sequence l_m with

$$\|f\chi_{A^m} \wedge l_m 1\| > \delta. \tag{5}$$

For a fixed k we choose $A_n^m \in \Sigma$, $n \in \mathbf{N}$, $m = 1, \dots, k$, with

$$\chi_{A^m} = [\chi_{A_n^m}], \qquad f\chi_{A^m} = [f\chi_{A_n^m}] \quad \text{and} \quad A_n^m \cap A_n^l = \varnothing$$

for $m, l \in \{1, \dots, k\}$, $m \neq l$, and all n. Since $\mathfrak{A}\text{-}\lim_n \mu(A_n^m) = \mu(A^m)$, $\mathfrak{A}\text{-}\lim \|f\chi_{A_n^m} \wedge l_m 1\| > \delta$ there is an index n_k such that for $m = 1, \dots, k$,

$$\mu(A_{n_k}^m) \leq 2\tilde{\mu}(A^m), \qquad \|f_{n_k}\chi_{A_{n_k}^m} \wedge l_m 1\| \geq \delta. \tag{6}$$

We will show now that f_{n_k} has no subsequence that splits. Otherwise, there were a subsequence $F_j = f_{n_{k_j}}$, an equi-integrable sequence $g_j \in X$, and a pairwise disjoint sequence h_j with $F_j = g_j + h_j$, $g_j \wedge h_j = 0$.

For notational convenience put also $B_j^m = A_{n_{k_j}}^m$. To obtain a contradiction we choose $\varepsilon > 0$ with

$$\mu(B) < \varepsilon \Rightarrow \sup_j \|g_j\chi_B\| < \delta/4. \tag{7}$$

Next we choose m such that $\tilde{\mu}(A^m) < \varepsilon/2$ and $j > m$ with

$$\|\chi_{\{h_j \neq 0\}}\| < \delta/4l_m. \tag{8}$$

Since $\mu(B_j^m) \leq 2\tilde{\mu}(A^m) \leq \varepsilon$ by (6), we obtain from (7) and (8) that

$$\|F_j\chi_{B_j^m} \wedge l_m 1\| \leq \|g_j\chi_{B_j^m}\| + \|h_j \wedge l_m 1\| \leq \delta/4 + l_m \cdot \delta/4l_m = \delta/2.$$

On the other hand, (6) implies the contradiction $\|F_j\chi_{B_j^m} \wedge l_m 1\| \geq \delta$.

(2) $\Rightarrow$ (1). For a bounded sequence $(f_n) \in X$, we can write $f = [f_n] = g + h$ with $g \in \tilde{X}$, $h \in \tilde{X}^\perp$. By Proposition 1.5 there are sequences (g_n) and (h_n) with $g = [g_n]$, $h = [h_n]$ such that (g_n) is equi-integrable in X and h_n goes to zero in measure. Hence $\|f_n - g - h_n\| \xrightarrow{\mathfrak{A}} 0$ and there is a subsequence n_k with $\|f_{n_k} - g_{n_k} - h_{n_k}\| \xrightarrow{k} 0$.

(2) $\Rightarrow$ (4). Assume to the contrary that there are $(f_i^n) \subset X$, $i = 1, \dots, n$, $n \in \mathbf{N}$, with the following properties

(i) $f_1^n, \dots, f_n^n$ are pairwise disjoint for all n, $\|f_i^n\| = 1$.
(ii) $f_1^n, \dots, f_n^n$ are $(1+\varepsilon)$-equivalent to the unit vector basis of l_∞^n.
(iii) $(d_{f_i^n})_{n=i, i+1, \dots}$ uniformly converges to some d_i on $[0, \infty)$.

Put $f_i^n = 0$ for $i > n$. Denote by g_i the $\tilde{X}$-component of $f_i = [f_i^n, n \in \mathbf{N}]$. Since

$$\begin{aligned}\|g_i \wedge m1\|_{\tilde{X}} &= \|f_i \wedge m1\| = \mathfrak{A}\text{-}\lim_n \|f_i^n \wedge m\| \\ &= 1\text{-}\lim_n d_{f_i^n}(m) = 1 - d_i(m)\end{aligned}$$

and $\lim_{t\to\infty} d_i(t) = 0$ we have $\|g_i\|_{\tilde{X}} = 1$. The g_i's are also pairwise disjoint by (i). We get from (ii) that $f = [\sum_{i=1}^n f_i^n] \in X_{\mathfrak{A}}$ with $\|f\| \leq 1 + \varepsilon$. If g is the $\tilde{X}$-component of f we have $g_i \leq g$ for all $i \in \mathbf{N}$ and the g_i's span l^∞

in $\widetilde{X}$. Therefore, $\widetilde{X}$ cannot be order continuous if l_n^∞'s embed equinormably into X.

(4) $\Rightarrow$ (3). Assume that c_0 embeds into $\widetilde{X}$, i.e., there is a pairwise disjoint sequence $f_i \in \widetilde{X}$, $\|f_i\| = 1$, which is $(1+\varepsilon)$-equivalent to the unit vector basis of c_0. For a fixed $k \in \mathbf{N}$ we choose representations $(f_i^n)_n$ of f_i for $i = 1, \dots, k$ with $f_i^n \wedge f_j^n = 0$ for $i \neq j$ and $n \in \mathbf{N}$. There is a $U_k \in \mathfrak{A}$ such that for all $n \in U_k$

$$\left\|\sum_{i=1}^k f_i^n\right\| \le 1+\varepsilon, \qquad \|f_i^n\| \ge 1-\varepsilon \quad \text{for } i = 1, \dots, k.$$

This implies already that $[f_1^n, \dots, f_k^n]$ is $(1+\varepsilon)$-isomorphic to l_∞^k for all $n \in U_k$. It follows from Proposition 1.3 and the equi-continuity (compare Proposition 1.7) of the functions $d_{f_i^n}$, $n \in \mathbf{N}$, that $\mathfrak{A}\text{-}\lim_n d_{f_i^n} = d_{f_i}$ with respect to uniform convergence. Therefore we can choose $n_k \in U_k$ with

$$\sup_{t>0} |d_{f_i^{n_k}}(t) - d_{f_i}(t)| < \tfrac{1}{k} \quad \text{for } i = 1, \dots, k.$$

(3) $\Rightarrow$ (2). A Banach lattice, which is not order continuous contains l^∞ [14, Theorem 5.14]. □

2.6. **Corollary.** *Every rearrangement invariant function space X which does not contain c_0, has the subsequence splitting property.*

Proof. c_0 is not contained in X iff every increasing sequence $f_n \le f_{n+1} \le \cdots$ with $\|f_n\| \le C$ has a sup in X and $f_n \nearrow \sup f_n$ in norm (e.g. [14, Proposition 5.15]). It follows now from Example 1.2.b that if X has this property, $\widetilde{X}$ has it too.

There is also a more direct proof of the corollary: By Example 1.2.b, we can assume that X is a rearrangement invariant space on $[0, 1]$. For a bounded sequence $f_n \in X$ we choose by Helley's theorem a subsequence of f_n^* (which we call again f_n^*) such that $f_n^* \to f$ pointwise to some measurable function f. By order continuity we have $f|_{[1/m,1]} \in X$ for all m and $\|f|_{[1/m,1]}\| \le \sup_n \|f_n\|$. Hence $f \in X$ by the Fatou property. For every m we choose n_m such that $\|(f_{n_m}^* \vee f - f)\chi_{[1/m,1]}\| \le \frac{1}{m}$. Then

$$g_m = f_{n_m}\chi_{\{|f_{n_m}| \le f_{n_m}^*(1/m)\}}$$

is equi-integrable in X and $h_m = f_{n_m} - g_m$ goes to zero in measure. □

2.7. **Corollary.** *X and X^* have the subsequence splitting property iff $\widetilde{X}$ is reflexive. In this case $\widetilde{X}^* = \widetilde{X^*}$.*

Proof. "$\Rightarrow$" Since $\widetilde{X}$ has order continuous norm, we have by [12, p. 29],

$$\widetilde{X}^* = \widetilde{X}^\times = \left\{ f \in L_1(\widetilde{\Omega}, \widetilde{\mu}) \middle| \int f \cdot g \, d\widetilde{\mu} < \infty \quad \text{for all } g \in \widetilde{X} \right\}.$$

We have $L_\infty(\widetilde{\Omega}, \widetilde{\mu}) \subset \widetilde{X^*} \subset \widetilde{X}^\times$ with $\|f\|_{\widetilde{X^*}} = \sup\{\int f \cdot g \, d\widetilde{\mu} : \|g\|_{\widetilde{X}} \le 1\}$.

Indeed, since $\widetilde{X^*}$ has order continuous norm, we can represent $f \in \widetilde{X^*}$ as $f = [f_n]$ with an equi-integrable sequence f_n in X^* (cf. Proposition 1.5). Choose $g_n \in X$ with $\|g_n\| = 1$ and $f_n(g_n) = \|f_n\|_{X^*}$. Put $[g_n] = \bar{g} + \underline{g}$ with $\bar{g} \in \widetilde{X}$, $\underline{g} = [\underline{g}_n] \in \widetilde{X}^{\perp}$ and note that $\mathfrak{A}\text{-}\lim f_n(\underline{g}_n) = 0$ since (f_n) is equi-integrable in X^*. Hence

$$\|f\|_{\widetilde{X^*}} = \mathfrak{A}\text{-}\lim \|f_n\| = \mathfrak{A}\text{-}\lim f_n(\bar{g}_n) \leq f(\bar{g}) \leq \|f\|_{\widetilde{X}^\times}.$$

The reverse inequality is clear.

For $f \in \widetilde{X}^\times$ we have $f \wedge n1 \in L_\infty(\widetilde{\mu}) \subset \widetilde{X^*}$ and $\|f \wedge n1\|_{\widetilde{X^*}} \leq \|f\|_{X^\times}$. Since $\widetilde{X^*}$ does not contain c_0, it follows from [12] 1.c.4 that $f = \sup_n f \wedge n1 \in \widetilde{X}^*$. Hence $\widetilde{X}^* = \widetilde{X^*}$ has also order continuous norm and it follows from [12] 1.c.5 that $\widetilde{X}$ is reflexive. "$\Leftarrow$" Use [12] 1.c.5. □

Next we characterize the SSP in terms of a 'uniform' order continuity property.

2.8. **Theorem.** *For a Banach lattice X, the following are equivalent*

(1) *X has the subsequence splitting property*

(2) *X has an equivalent lattice norm $|\|\cdot\||$ such that for every $d \in C(0,\infty)$ with $0 < d(t) \to 0$ for $t \to \infty$ we have*

$$\sup\{|\|\chi_A f\||: f \in X \text{ with } d_f \leq d\} \to 0 \text{ for } \mu(A) \to 0.$$

(3) *X has an equivalent lattice norm $|\|\cdot\||$ such that a set $M \subset X$ is equi-integrable in X if and only if $\{d_f : f \in M\}$ is a compact subset of $C(0,\infty)$ with the sup-norm.*

Remark. The typical renorming we use in (2) and (3) is

$$(9)\qquad |\|f\|| = \|f\| + \sum_{k=m}^{\infty} 2^{-k}\|f\|_k, \qquad \|f\|_k = \sup\left\{\|\chi_A f\| : \mu(A) \leq \frac{1}{k}\right\}.$$

Observe that $X = (L_2[-1,0] \oplus L_2[0,1])_\infty$ has SSP but does not satisfy (2) or (3). Indeed, if we choose $f_n \in X$ such that the $f_n|_{(-1,0]}$ are pairwise disjoint with $\|f_n|_{[-1,0]}\| = 1$ and the $f_n|_{[0,1]}$ are the Rademacher functions then $d_{f_n}(t) = 0$ for $t > 1$, although f_n is not equi-integrable.

Proof. (1) $\Rightarrow$ (3) We observed already in 1.7 that for an equi-integrable set M the set $\{d_f, f \in M\}$ is compact in $C[0,\infty)$. To prove the converse we may restrict ourselves to sets M of positive functions. We use the renorming (9). Let $M \subset (X, |\|\cdot\||)$ be such that $\{d_f : f \in M\}$ is compact in $C[0,\infty)$. For $f_n \in M$ with $t \to |\|f_n \wedge t1\||$ converging uniformly on $[0,\infty)$ to a function $d \in C[0,\infty)$, we have to show that (f_n) is equi-integrable. Otherwise, by (1) there are a subsequence of f_n (which we denote again by f_n), an equi-integrable sequence g_n and a pairwise disjoint sequence (h_n) with

$$f_n = h_n + g_n, \qquad h_n \wedge g_n = 0, \qquad \|h_n\| \geq \delta > 0.$$

By Proposition 1.7 we can assume that $t \to |\,\|g_n \wedge t1\|\,|$ also converges uniformly to d. On the other hand, there is an $M < \infty$ and $N \in \mathbf{N}$ such that

$$\|g_n\|_m = \sup\{\|\chi_A g_n\|: \mu(A) \leq \tfrac{1}{m}\} \leq \delta/4 \quad \text{for all } n \text{ and } m \geq M,$$
$$\|h_n\|_M \geq 3\delta/4 \quad \text{for all } n \geq N.$$

For $n \geq N$ and all $t \in (0, \infty)$ we obtain

$$\begin{aligned} |\,\|f_n \wedge t1\|\,| &\geq \|g_n \wedge t\| + \sum_{m=1}^{M-1} 2^{-m}\|g_n \wedge t1\|_m + \sum_{m=M}^{\infty} 2^{-m}\|h_n \wedge t1\|_m \\ &\geq |\,\|g_n \wedge t1\|\,| - \sum_{m=M}^{\infty} 2^{-m}\|g_n\|_m + 2^{-M}(\|h_n\|_M - \|h_n - h_n \wedge t1\|_M) \\ &\geq |\,\|g_n \wedge t1\|\,| + 2^{-M}(\delta/2 - \|h_n - h_n \wedge t\|). \end{aligned}$$

Hence for every $n \geq N$ there is a $t \in (0, \infty)$ with

$$|\,\|f_n \wedge t1\|\,| - |\,\|g_n \wedge t1\|\,| \geq 2^{-M-2}\delta > 0.$$

But we observed earlier that this difference should go to zero uniformly.

(3) $\Rightarrow$ (2). Choose an equivalent lattice norm on X for which (3) holds and observe that the set $M = \{d_f : f \in X; d_f \leq d\}$ is compact in $C[0, \infty]$ by the same argument we used in the proof of Proposition 1.7.

(2) $\Rightarrow$ (1). Choose an equivalent lattice norm on X for which (2) holds.

Let (f_n) be a bounded, positive sequence in X. Since the functions $s \to \|f_n \wedge s1\|$, $n \in \mathbf{N}$, are equi-continuous on every finite interval $[0, t]$, we can choose a subsequence of f_n (call it again (f_n)) such that the functions $s \to \|f_n \wedge s1\|$, $n \in \mathbf{N}$, converge uniformly on every $[0, m]$, $m \in \mathbf{N}$. Put $C = \sup_{t>0} \lim_n \|f_n \wedge t\|$.

For every $m \in \mathbf{N}$ we choose n_m and t_m with $n_m > n_{m-1}$, $t_m < t_{m+1} \to \infty$ such that

$$C + \tfrac{1}{m} \geq \|f_n \wedge t_m\| \geq C - \tfrac{1}{m} \quad \text{for all } n \geq n_m. \tag{10}$$

Put now $g_m = f_{n_m} \wedge t_m 1$, $h_m = f_{n_m} - f_{n_m} \wedge t_m 1$ and $d(t) = C - \inf_m \|g_m \wedge t\|$. d is decreasing and assume for a moment that $\dim_{t\to\infty} d(t) = 0$. Since $\|g_m\| \to C$ by (10) we can assume that $\|g_m\| = C$. Then $d_{g_m}(t) \leq d(t)$ for $t \geq 0$ and it follows from (2) that (g_m) is equi-integrable. Observe that $\{h_m \neq 0\} \subset \{f_{n_m} \geq t_m\} = A_m$ and that $\mu(A_m) \to 0$ for $m \to \infty$ since $t_m \to \infty$ and (f_n) is bounded in $L_1(\mu)$. Now we obtain the required splitting for $f_{n_m} = g_m + h_m$ by applying a disjointification procedure to (h_m).

It remains to show that $\lim_{t\to\infty} d(t) = 0$. Otherwise there is a $l \in \mathbf{N}$ and a subsequence (m_k) with $\|g_{m_k} \wedge k\| \leq C - 2l^{-1}$.

Since $\|g_m\| = C$ we have $m_k \to \infty$ for $k \to \infty$. Therefore, for a k with $t_{m_k} \wedge k \geq t_l$ and $n_{m_k} \geq n_l$ we get from (10) that

$$\|g_{m_k} \wedge k\| \geq \|f_{n_{m_k}} \wedge t_l\| \geq C - l^{-1}.$$

This contradiction completes the proof. □

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