MULTIPLIERS ON WEIGHTED L_p -SPACES OVER LOCALLY COMPACT VILENKIN GROUPS

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ABSTRACT. Let G be a locally compact Vilenkin group. We prove a multiplier theorem for power-weighted $L_p(G)$ -spaces which extends an earlier result of this kind, due to Onneweer.

1. INTRODUCTION

In a recent paper [4] T. Kitada obtained some new theorems for multipliers on weighted Hardy spaces defined on certain totally disconnected groups G that are the locally compact analogue of the Vilenkin groups. One of Kitada's results that is of interest here gives a sufficient condition for a function in $L_\infty(\Gamma)$, Γ being the dual group of G , to be a multiplier from a power-weighted Hardy space $H_{1,\alpha}(G)$ to the corresponding weighted Lebesgue space $L_{1,\alpha}(G)$. As an immediate corollary to this result Kitada derived a theorem proved earlier by one of the present authors [6, Theorem 1]. Combining some of the results obtained by Kitada with techniques developed by Bagby and Kurtz in [1] we prove here a multiplier theorem for $L_{p,\alpha}(G)$ -spaces that extends Theorem 1 in [6] considerably.

2. DEFINITIONS AND NOTATION

Throughout this paper G will denote a locally compact Abelian group containing a strictly increasing sequence of open compact subgroups $(G_n)_{n=-\infty}^\infty$ such that

- (i) $\sup_n \{\text{order } G_n/G_{n+1}\} < \infty$, and
- (ii) $\bigcup_{-\infty}^\infty G_n = G$ and $\bigcap_{-\infty}^\infty G_n = \{0\}$.

Such groups are the locally compact analogue of the so-called Vilenkin groups. Examples of such groups are given in [2, §4.1.2]. Additional examples are the p -adic numbers and, more general, the additive group of a local field, see [8].

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Let Γ be the dual group of G and for each $n \in \mathbf{Z}$ let $\Gamma_n = \{\gamma \in \Gamma; \gamma(x) = 1 \text{ for all } x \in G_n\}$. Then, see [2, §4.1.4], $(\Gamma_n)_{-\infty}^{\infty}$ is a strictly decreasing sequence of open compact subgroups of Γ such that (i) $\text{order}(\Gamma_{n+1}/\Gamma_n) = \text{order}(G_n/G_{n+1})$ and (ii) $\bigcup_{-\infty}^{\infty} \Gamma_n = \Gamma$ and $\bigcap_{-\infty}^{\infty} (\Gamma_n) = \{\gamma_0\}$, where $\gamma_0(x) = 1$ for all $x \in G$.

If we choose Haar measures μ on G and λ on Γ so that $\mu(G_0) = \lambda(\Gamma_0) = 1$ then $\mu(G_n) = (\lambda(\Gamma_n))^{-1} := (m_n)^{-1}$ for each $n \in \mathbf{Z}$. There exists a metric d on G defined by $d(x, x) = 0$ and $d(x, y) = (m_n)^{-1}$ if $x - y \in G_n \setminus G_{n+1}$. For $x \in G$ we set $|x| = d(x, 0)$ and for each $\alpha \in \mathbf{R}$ we define the function v_α on G by $v_\alpha(x) = |x|^\alpha$; the corresponding measure $v_\alpha d\mu$ will also be denoted by $d\mu_\alpha$. The Lebesgue function spaces on G with respect to the measures $d\mu_\alpha$ will be denoted by $L_{p,\alpha}(G)$ or $L_{p,\alpha}$ and for $f \in L_{p,\alpha}$, $1 \leq p \leq \infty$ and $\alpha \in \mathbf{R}$, we set

$$\|f\|_{p,\alpha} = \left(\int_G |f(x)|^p d\mu_\alpha \right)^{1/p},$$

with the usual modification if $p = \infty$. If $\alpha = 0$ we write, as usual, L_p and $\|f\|_p$ instead of $L_{p,0}$ and $\|f\|_{p,0}$.

For p with $1 \leq p \leq \infty$ we denote its conjugate by p' , thus $p + p' = pp'$. For an arbitrary set A we denote its indicator function by ξ_A . The symbols $^{\wedge}$ and $^{\vee}$ will be used to denote the Fourier transform and inverse Fourier transform, respectively. The letter C will be used to denote a constant whose value may change from occurrence to occurrence.

For a definition of the spaces of test functions and distributions on G or Γ , see [6 or 8]; these spaces will be denoted by $\mathcal{S}(G)$, $\mathcal{S}'(G)$, $\mathcal{S}(\Gamma)$ and $\mathcal{S}'(\Gamma)$ and we can extend the Fourier and inverse Fourier transform to $\mathcal{S}'(G)$ and $\mathcal{S}'(\Gamma)$ in the standard way and the usual properties hold, see [8] for details. Also, if $\alpha > -1$ and $1 \leq p < \infty$ then $\mathcal{S}(G)$ is dense in $L_{p,\alpha}(G)$.

We now turn to the definition of the weighted Hardy spaces $H_{1,\alpha}(G)$ and the definition of certain multipliers, cf. [4].

Definition 1. Let $\alpha \in \mathbf{R}$. A function a on G is a $(1, \infty)_\alpha$ atom if there exists a set $I = x + G_n$ ($x \in G$, $n \in \mathbf{Z}$) so that

- (i) $\text{supp}(a) \subset I$,
- (ii) $\|a\|_\infty \leq (\mu_\alpha(I))^{-1}$ and,
- (iii) $\int_G a d\mu = 0$.

Definition 2. Let $\alpha \in \mathbf{R}$. A function f on G belongs to $H_{1,\alpha}(G) = H_{1,\alpha}$ if f can be decomposed as $f(x) = \sum_{k=1}^{\infty} \lambda_k a_k(x)$, where each a_k is a $(1, \infty)_\alpha$ atom and $\sum_{k=1}^{\infty} |\lambda_k| < \infty$. If $f \in H_{1,\alpha}$ we set $\|f\|_{H_{1,\alpha}} = \inf \sum_{k=1}^{\infty} |\lambda_k|$, where the infimum is taken over all decompositions of f in $(1, \infty)_\alpha$ atoms.

Definition 3. Let X and Y denote any of the spaces $H_{1,\alpha}$ or $L_{p,\alpha}$ with $\alpha \in \mathbf{R}$ and $1 \leq p < \infty$. For $\phi \in L_\infty(\Gamma)$ and $f \in \mathcal{S}(G)$ let $T_\phi f := (\phi \hat{f})^\vee$. A function

$\phi \in L_\infty(\Gamma)$ is an (X, Y) multiplier (or a multiplier on X if $X = Y$) if there exists a $C > 0$ so that for all $f \in \mathcal{S}(G) \cap X$ we have $\|T_\phi f\|_Y \leq C\|f\|_X$.

The following definition is a slight modification of a definition given in [6].

Definition 4. Let $\phi \in L_\infty(\Gamma)$ have support in Γ_k for some $k \in \mathbf{Z}$.

(i) ϕ is said to satisfy condition $C(r)$, $1 < r < \infty$, if there exist constants $C, \varepsilon > 0$ so that for all $l \in \mathbf{Z}$ and all $n < l$ we have

$$\sup \left\{ \left(\int_{G_n \setminus G_{n+1}} |\phi^\vee(x-y) - \phi^\vee(x)|^r d\mu(x) \right)^{1/r}; y \in G_l \right\} \leq C(m_n)^{\varepsilon+1/r'}(m_l)^{-\varepsilon}.$$

(ii) ϕ is said to satisfy condition $C(1)$ if there exists $C > 0$ so that for all $l \in \mathbf{Z}$ we have

$$\sup \left\{ \int_{G \setminus G_l} |\phi^\vee(x-y) - \phi^\vee(x)| d\mu(x); y \in G_l \right\} \leq C.$$

It follows immediately from Hölder's inequality that if ϕ satisfies condition $C(r)$ for some $r \in [1, \infty)$ then ϕ satisfies $C(s)$ for all s with $1 \leq s \leq r$.

For ease of reference later on we now state two of the main results of Kitada; the first is an interpolation theorem for weighted Hardy and Lebesgue spaces on G [4, Theorem C], the second is an $(H_{1,\alpha}, L_{1,\alpha})$ multiplier theorem [4, Theorem 1].

Theorem K1. Let $-1 < \alpha \leq 0$ and $1 < p_1 \leq \infty$. If T is a sublinear operator of weak type $(1, 1)$ on $H_{1,\alpha}$ and of weak type on $L_{p_1,\alpha}$ then T is of type (p, p) on $L_{p,\alpha}$ for all p with $1 < p < p_1$.

Theorem K2. Let $\phi \in L_\infty(\Gamma)$ have support in Γ_k for some $k \in \mathbf{Z}$ and assume that ϕ satisfies condition $C(r)$ for some $r \in [1, \infty)$. Then ϕ is an $(H_{1,\alpha}, L_{1,\alpha})$ multiplier for each α with $-1/r' < \alpha \leq 0$.

As was observed in [4], Theorem K2 implies the following multiplier theorem of Onneweer, cf. [4, Corollary] or [6, Theorem 1].

Theorem O1. Let $\phi \in L_\infty(\Gamma)$ and assume that for some $r \in (1, \infty)$ and all $k \in \mathbf{Z}$ the functions $\phi_k := \phi \chi_{\Gamma_k}$ satisfy condition $C(r)$ with constants C and ε independent of $k \in \mathbf{Z}$. If ϕ is a multiplier on L_{2,α_0} for some α_0 with $-1/r' < \alpha_0 < 1/r'$ then ϕ is a multiplier on $L_{p,\alpha}$ for all $p \in (1, \infty)$ and all α such that $-|\alpha_0| \leq \alpha \leq (p-1)|\alpha_0|$.

In §3 we shall prove the following result which is an improvement of the theorem just stated.

Theorem 1. Let $\phi \in L_\infty(\Gamma)$ and assume that for some $r \in (1, \infty)$ the functions $\phi_k = \phi \chi_{\Gamma_k}$ satisfy condition $C(r)$ with constants C and ε independent of

$k \in \mathbb{Z}$. Then ϕ is a multiplier on $L_{p,\alpha}$ for $(p, \alpha) \in D = \bigcup_{i=1}^4 D_i$, where

$$\begin{aligned} D_1 &= \{(p, \alpha); 1 < p \leq r \text{ and } 0 \leq \alpha < p - 1\}, \\ D_2 &= \{(p, \alpha); r \leq p < \infty \text{ and } 0 \leq \alpha < p/r'\}, \\ D_3 &= \{(p, \alpha); 1 < p \leq r' \text{ and } -p/r' < \alpha \leq 0\}, \\ D_4 &= \{(p, \alpha); r' < p < \infty \text{ and } -1 < \alpha \leq 0\}. \end{aligned}$$

3. PROOF OF THEOREM 1

Before turning to the proof of Theorem 1, we first introduce some additional notation. The concepts introduced here in the context of the groups G are analogous to well-known concepts on $\mathbb{R}^n$. For a locally integrable function f on G we define the Hardy-Littlewood maximal function Mf and the Fefferman-Stein "sharp" function $f^\#$ by

$$Mf(x) = \sup_n (\mu(x + G_n))^{-1} \int_{x+G_n} |f(y)| d\mu$$

and

$$f^\#(x) = \sup_n (\mu(x + G_n))^{-1} \int_{x+G_n} |f(y) - f_{x+G_n}| d\mu,$$

where

$$f_{x+G_n} = (\mu(x + G_n))^{-1} \int_{x+G_n} f(y) d\mu.$$

The Muckenhoupt classes A_p , $1 \leq p \leq \infty$, of weight functions are defined on G as on $\mathbb{R}^n$, replacing the cubes Q in the definitions of Muckenhoupt by the cosets of the subgroups G_n of G , $n \in \mathbb{Z}$, see [3, Chapter 4]. It is easy to see that the functions $v_\alpha(x) = |x|^\alpha$ on G belong to A_p whenever $-1 < \alpha < p - 1$. If f is a measurable function on G , the rearrangement of f with respect to the weight function w on G is, for $t > 0$, defined by

$$f_w^*(t) = \inf\{\lambda > 0; w(\{x \in G; |f(x)| > \lambda\}) \leq t\},$$

where $w(A) = \int_A w(x) d\mu$ for a measurable subset A of G .

In [1] Bagby and Kurtz proved two covering lemmas for open sets $\Omega \subset \mathbb{R}^n$ and they used these lemmas to estimate, for a given $f \in L_{1,\text{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal function Mf in terms of the sharp function $f^\#$. The following theorem gives an analogue on G of Theorem 4.2 in [1].

Theorem 2. *Let $f \in L_{1,\text{loc}}(G)$ and $w \in A_\infty$. Then there exists a $C > 0$ so that for all $t > 0$,*

$$(Mf)_w^*(t) \leq (Mf)_w^*(2t) + C(f^\#)_w^*(2t).$$

We precede the proof of Theorem 2 by two lemmas.

Lemma 1. Let $f \in L_{1,\text{loc}}(G)$, $w \in A_\infty$ and $t > 0$. Let $A = \{y \in G; Mf(y) > (Mf)_w^*(2t)\}$. Then there exists a (possibly finite) sequence of mutually disjoint cosets $(y_i + G_{\alpha_i})$ such that

$$(1) \quad A = \bigcup_i y_i + G_{\alpha_i},$$

$$(2) \quad (\mu(y_i + G_{\alpha_i}))^{-1} \int_{y_i + G_{\alpha_i}} |f(x)| d\mu > (Mf)_w^*(2t),$$

$$(3) \quad (\mu(y_i + G_n))^{-1} \int_{y_i + G_n} |f(x)| d\mu \leq (Mf)_w^*(2t) \text{ for every } n < \alpha_i.$$

Proof. Let (β_n) be a monotone decreasing sequence converging to $(Mf)_w^*(2t)$. Then

$$A \subset \bigcup_{n=1}^{\infty} \{y; Mf(y) > \beta_n\} := \bigcup_{n=1}^{\infty} A_n,$$

where $A_n \subset A_{n+1}$ for each $n \in \mathbb{N}$. Also, it follows immediately from the definition of $(Mf)_w^*(2t)$ that $w(A_n) \leq 2t$ for each $n \in \mathbb{N}$ and, hence, $w(A) \leq 2t$. For any $y \in A$ there exists $n \in \mathbb{Z}$ so that

$$\begin{aligned} (\mu(y + G_n))^{-1} \int_{y + G_n} |f(x)| d\mu &> Mf(y) - (Mf(y) - (Mf)_w^*(2t))/2 \\ &> (Mf)_w^*(2t). \end{aligned}$$

Thus, if

$$A(y) := \left\{ k \in \mathbb{Z}; (\mu(y + G_k))^{-1} \int_{y + G_k} |f(x)| d\mu > (Mf)_w^*(2t) \right\},$$

then $A(y)$ is nonempty. Also, $A(y)$ is bounded from below, for otherwise we would have $A = G$ and hence $w(A) = \infty$, since $w \in A_\infty$. Let $\alpha(y) = \min\{k; k \in A(y)\}$. Then $y + G_{\alpha(y)} \subset A$ and $y + G_n \not\subset A$ for all $n < \alpha(y)$. Denote the at most countably many different cosets in $\{y + G_{\alpha(y)}, y \in A\}$ by $y_i + G_{\alpha_i}$; then these sets clearly satisfy (1), (2) and (3).

Using a similar proof as for Lemma 1, we also have the following.

Lemma 2. Let $f \in L_{1,\text{loc}}(G)$, $w \in A_\infty$ and $t > 0$. Let $B = \{z \in G; f^\#(z) > (f^\#)_w^*(2t)\}$. Then there exists a (possibly finite) sequence of mutually disjoint cosets $(z_i + G_{\beta_i})$ such that

$$(4) \quad B = \bigcup_i z_i + G_{\beta_i},$$

$$(5) \quad (\mu(z + G_{\beta_i}))^{-1} \int_{z + G_{\beta_i}} |f(x) - f_{z + G_{\beta_i}}| d\mu > (f^\#)_w^*(2t),$$

$$(6) \quad (\mu(z + G_n))^{-1} \int_{z + G_n} |f(x) - f_{z + G_n}| d\mu \leq (f^\#)_w^*(2t) \text{ for all } n < \beta_i.$$

Proof of Theorem 2. Fix $t > 0$ and let $\Omega = A \cup B$, with $A = \bigcup y_i + G_{\alpha_i}$ and $B = \bigcup z_i + G_{\beta_i}$ as in Lemmas 1 and 2. Then $w(A) \leq 2t$ and $w(B) \leq 2t$ so

that $w(\Omega) < 5t$. Select those $y_i + G_{\alpha_i}$ in A that are disjoint from B and those $z_j + G_{\beta_j}$ in B that are disjoint from A . Furthermore, if $y_i + G_{\alpha_i} \cap z_j + G_{\beta_j} \neq \emptyset$ for some i and j then one of these sets is contained in the other and we select the larger coset. Denote the cosets thus selected by $x_i + G_{n_i}$. Then

$$(7) \quad \Omega = \bigcup x_i + G_{n_i},$$

and for all i and all $l < n_i$ we have

$$(8) \quad (\mu(x_i + G_l))^{-1} \int_{x_i + G_l} |f(x)| d\mu \leq (Mf)_w^*(2t)$$

and

$$(9) \quad (\mu(x_i + G_l))^{-1} \int_{x_i + G_l} |f(x) - f_{x_i + G_l}| d\mu \leq (f^\#)_w^*(2t).$$

To simplify the notation, set $Q_i = x_i + G_{n_i}$ and $\tilde{Q}_i = x_i + G_{n_i-1}$. Take $\lambda \in \mathbb{R}$ so that $\lambda > (Mf)_w^*(2t)$ and let $E_i = \{x \in Q_i; Mf(x) > \lambda\}$. If $x \in E_i$ then there exists an $l \in \mathbb{Z}$ so that

$$(\mu(x + G_l))^{-1} \int_{x + G_l} |f(t)| d\mu > \lambda.$$

It follows from (8) that $l \geq n_i$. Thus the set

$$\left\{ n \in \mathbb{Z}; (\mu(x + G_n))^{-1} \int_{x + G_n} |f(t)| d\mu > \lambda \right\}$$

is bounded below by n_i . Arguing as in the proof of Lemma 1, we can express E_i as $E_i = \bigcup T_{j(i)}$, where $T_{j(i)} = y_{j(i)} + G_{n_{j(i)}}$ for some $y_{j(i)} \in E_i$, and each $T_{j(i)}$ and $\tilde{T}_{j(i)} := y_{j(i)} + G_{n_{j(i)}-1}$ satisfy

$$(10) \quad (\mu(T_{j(i)}))^{-1} \int_{T_{j(i)}} |f(t)| d\mu > \lambda,$$

$$(11) \quad (\mu(\tilde{T}_{j(i)}))^{-1} \int_{\tilde{T}_{j(i)}} |f(t)| d\mu \leq \lambda.$$

Therefore, it follows from (10) and (8) that

$$\begin{aligned} (\lambda - (Mf)_w^*(2t))\mu(T_{j(i)}) &\leq \int_{T_{j(i)}} |f(t)| d\mu - \int_{T_{j(i)}} |f_{\tilde{Q}_i}| d\mu \\ &\leq \int_{T_{j(i)}} |f(t) - f_{\tilde{Q}_i}| d\mu. \end{aligned}$$

Summing over all $j(i)$ we see that

$$\begin{aligned} (12) \quad (\lambda - (Mf)_w^*(2t))\mu(E_i) &\leq \sum_{j(i)} \int_{T_{j(i)}} |f(t) - f_{\tilde{Q}_i}| d\mu \\ &\leq \int_{\tilde{Q}_i} |f(t) - f_{\tilde{Q}_i}| d\mu \\ &\leq \mu(\tilde{Q}_i)(f)_w^*(2t) \quad \text{by (9)} \\ &\leq L\mu(Q_i)(f)_w^*(2t), \end{aligned}$$

where $L = \sup_n (\mu(G_n)/\mu(G_{n+1}))$. Since $w \in A_\infty$, there exists an $\varepsilon > 0$ so that for any set $Q = x + G_n$, if $E \subset Q$ and $\mu(E) \leq \varepsilon\mu(Q)$ then $w(E) \leq w(Q)/5$. Choosing

$$\lambda = (L/\varepsilon)(f^\#)_w^*(2t) + (Mf)_w^*(2t)$$

in inequality (12) we obtain $\mu(E_i) \leq \varepsilon\mu(Q_i)$. Consequently,

$$\begin{aligned} w(\{x; Mf(x) > (L/\varepsilon)(f^\#)_w^*(2t) + (Mf)_w^*(2t)\}) &\leq \sum_i w(E_i) \\ &\leq \frac{1}{5} \sum_i w(Q_i) \leq t, \end{aligned}$$

that is,

$$(Mf)_w^*(t) \leq C(f^\#)_w^*(2t) + (Mf)_w^*(2t),$$

where $C = L/\varepsilon$. This completes the proof of Theorem 2.

The following corollary will be used later on.

Corollary 1. *Let f be a continuous function on G that vanishes at infinity and let $w \in A_\infty$. If $1 < p < \infty$ we have $\|Mf\|_{p,w} \leq C\|f^\#\|_{p,w}$.*

Proof. As was observed by Bagby and Kurtz in [1, Corollary 4.3(b)], it is sufficient to show that $\lim_{t \rightarrow \infty} (Mf)_w^*(t) = 0$. Take any $\varepsilon > 0$. There exists a subgroup G_i of G such that $|f(x)| < \varepsilon/3$ for all $x \notin G_i$. Let $E = \{x; x \notin G_i \text{ and if } G_i \cap (x + G_n) \neq \phi \text{ for some } n \in \mathbf{Z} \text{ we have } \mu(G_i)/\mu(x + G_n) < \varepsilon/3(\|f\|_\infty + 1)\}$. Clearly, $w(E) = \infty$ and $w(G \setminus E) < \infty$. Also, for $x \in E$ and any $j \in \mathbf{Z}$ we have

$$(\mu(x + G_j))^{-1} \int_{x+G_j} |f(y)| d\mu \leq \begin{cases} \varepsilon/3 & \text{if } (x + G_j) \cap G_i = \phi, \\ 2\varepsilon/3 & \text{if } (x + G_j) \cap G_i \neq \phi. \end{cases}$$

Hence we have $Mf(x) < \varepsilon$ for all $x \in E$ and this implies that $\lim_{t \rightarrow \infty} (Mf)_w^*(t) = 0$.

The following is the analogue on G of a result of Kurtz and Wheeden for functions on $\mathbf{R}^n$, cf. [5, pp. 353–354].

Theorem 3. *Let $\phi \in L_\infty(\Gamma)$ have its support in Γ_k for some $k \in \mathbf{Z}$ and suppose that ϕ satisfies condition $C(r)$ for some $r \in (1, \infty)$. Then there exists a $C > 0$ so that for all $f \in \mathcal{S}(G)$ and $x \in G$ we have $(T_\phi f)^\#(x) \leq C(M_r f)(x)$, where*

$$(M_r f)(x) = (M(|f|^{r'}))(x))^{1/r'}$$

and where C is independent of k .

Proof. Fix $x_0 \in G$. For each $s \in \mathbf{Z}$ let $I_s = x_0 + G_s$. For $f \in \mathcal{S}(G)$ and a fixed $s \in \mathbf{Z}$ express f as $f = f_s + \sum_{j=-\infty}^{s-1} f^j$, where $f_s = f \xi_{I_s}$ and $f^j = f_j - f_{j+1} = f \xi_{I_j \setminus I_{j+1}}$ for $-\infty < j < s$. For $y \in I_s$ we have

$$T_\phi f(y) = (\phi^\vee * f_s)(y) + \sum_{j=-\infty}^{s-1} (\phi^\vee * f^j)(y).$$

First we observe that, by Hölder's inequality,

$$\begin{aligned} (\mu(I_s))^{-1} \int_{I_s} |(\phi^\vee * f_s)(y)| d\mu &\leq m_s \left(\int_{I_s} |(\phi^\vee * f_s)(y)|^{r'} d\mu \right)^{1/r'} \left(\int_{I_s} 1 d\mu \right)^{1/r} \\ &\leq C(m_s)^{1/r'} \|f_s\|_{r'} \\ &\leq C(M_{r'}f)(x_0), \end{aligned}$$

where the second inequality follows from the fact that ϕ is a multiplier of $L_{r'}$, as follows from Theorem O1.

Next, for each j with $-\infty < j \leq s-1$ we have

$$\begin{aligned} (\phi^\vee * f^j)(y) &= (\phi^\vee * f^j)(x_0) + ((\phi^\vee * f^j)(y) - (\phi^\vee * f^j)(x_0)) \\ &= A_j + B_j. \end{aligned}$$

It follows from Hölder's inequality that

$$\begin{aligned} |B_j| &\leq \int_{I_j \setminus I_{j+1}} |\phi^\vee(y-z) - \phi^\vee(x_0-z)| |f^j(z)| d\mu(z) \\ &\leq \left(\int_{I_j \setminus I_{j+1}} |\phi^\vee(y-z) - \phi^\vee(x_0-z)|^r d\mu(z) \right)^{1/r} \cdot \left(\int_{I_j \setminus I_{j+1}} |f^j(z)|^{r'} d\mu \right)^{1/r'}. \end{aligned}$$

Substituting $t = x_0 - z$ in the first integral we see that it is equal to

$$\left(\int_{G_j \setminus G_{j+1}} |\phi^\vee(y - x_0 + t) - \phi^\vee(t)|^r d\mu(t) \right)^{1/r} \leq C(m_j)^{\varepsilon+1/r'} (m_s)^{-\varepsilon},$$

because $y - x_0 \in G_s$ and ϕ satisfies condition $C(r)$. Therefore, we see that

$$\begin{aligned} |B_j| &\leq C(m_j)^\varepsilon (m_s)^{-\varepsilon} \left(m_j \int_{I_j} |f(z)|^{r'} d\mu \right)^{1/r'} \\ &\leq C(m_j/m_s)^\varepsilon (M_{r'}f)(x_0). \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} m_s \int_{I_s} \left| (\phi^\vee * f)(y) - \sum_{j=-\infty}^{s-1} A_j \right| d\mu \\ \leq m_s \int_{I_s} |(\phi^\vee * f_s)(y)| d\mu + m_s \sum_{j=-\infty}^{s-1} \int_{I_s} |(\phi^\vee * f^j)(y) - A_j| d\mu \\ \leq C(M_{r'}f)(x_0) + C \sum_{j=-\infty}^{s-1} (m_j)^\varepsilon (m_s)^{-\varepsilon} (M_{r'}f)(x_0) \\ \leq C(M_{r'}f)(x_0). \end{aligned}$$

Since s is arbitrary we may conclude that $(T_\phi f)^\#(x_0) \leq C(M_{r'}f)(x_0)$, where C is independent of $k \in \mathbb{Z}$.

Combining the foregoing results we can prove the following.

Corollary 2. Let $\phi \in L_\infty(\Gamma)$ have support in Γ_k for some $k \in \mathbb{Z}$ and assume that ϕ satisfies condition $C(r)$ for some $r \in (1, \infty)$. Then ϕ is a multiplier on $L_{p,\alpha}$ for all p, α with $r' < p < \infty$ and $-1 < \alpha < p/r' - 1$.

Proof. Take any $f \in \mathcal{S}(G)$. Since $\alpha > -1$ we have $v_\alpha \in A_\infty$. Also, $T_\phi f = \phi^\vee * f$ is a continuous function vanishing at infinity. Thus, applying Corollary 1 to obtain the second inequality and Theorem 3 for the third inequality, we obtain

$$\begin{aligned} \|T_\phi f\|_{p,\alpha} &\leq M \|(T_\phi f)\|_{p,\alpha} \\ &\leq C \|(T_\phi f)^\#\|_{p,\alpha} \\ &\leq C \|M_{r'} f\|_{p,\alpha} \\ &= C \|M(|f|^{r'})\|_{p/r',\alpha}^{1/r'} \\ &\leq C \| |f|^{r'} \|_{p/r',\alpha}^{1/r'} = C \|f\|_{p,\alpha}, \end{aligned}$$

where the last inequality holds because $p/r' > 1$ and $v_\alpha \in A_{p/r'}$. Thus, we see that ϕ is a multiplier on $L_{p,\alpha}$.

Proof of Theorem 1. We first observe that in order to show that $\phi \in L_\infty(\Gamma)$ is a multiplier on $L_{p,\alpha}$ it is sufficient to show that each $\phi_k = \phi|_{\Gamma_k}$ is a multiplier on $L_{p,\alpha}$ with multiplier norm independent of k , see Remark 1 in [6] for details. It follows immediately from Corollary 2 that each ϕ_k is a multiplier on $L_{p,\alpha}$ for $(p, \alpha) \in D_4$ and with multiplier norms independent of k ; therefore, ϕ is a multiplier on $L_{p,\alpha}$ for $(p, \alpha) \in D_4$. Next, Theorem K1 and Theorem K2 imply that ϕ is a multiplier on $L_{p,\alpha}$ for $(p, \alpha) \in D_5$, where $D_5 = \{(p, \alpha); 1 < p < \infty \text{ and } -1/r' < \alpha \leq 0\}$. Furthermore, it is easy to see that if $(p_0, \alpha_0) \in D_3 \setminus D_5$ then there exist $(p_1, \alpha_1) \in D_5$ and $(p_2, \alpha_2) \in D_4$ so that for some $\theta \in (0, 1)$ we have

$$\frac{1}{p_0} = \theta \frac{1}{p_1} + (1 - \theta) \frac{1}{p_2} \quad \text{and} \quad \frac{\alpha_0}{p_0} = \theta \frac{\alpha_1}{p_1} + (1 - \theta) \frac{\alpha_2}{p_2}.$$

Thus, Stein's interpolation theorem for weighted L_p -spaces [7, Theorem 2] implies that ϕ is a multiplier on L_{p_0,α_0} . Finally, the duality theorem for multipliers on $L_{p,\alpha}$ -spaces [6, Proposition 1] implies that ϕ is a multiplier on $L_{p,\alpha}$ for $(p, \alpha) \in D_1 \cup D_2$. This completes the proof of Theorem 1.

We conclude by stating a corollary of Theorem 1 which is a strengthened version of Corollary 1 in [6].

Corollary 3. Let $\phi \in L_\infty(\Gamma)$ and assume that for all $k \in \mathbb{Z}$ and all $r \in (1, \infty)$ the functions $\phi_k = \phi|_{\Gamma_k}$ satisfy condition $C(r)$ with constants C and ε independent of k . Then ϕ is a multiplier on $L_{p,\alpha}$ for all (p, α) with $1 < p < \infty$ and $-1 < \alpha < p - 1$.

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