ON WEAKLY STATIONARY SETS

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ABSTRACT. We consider whether every weakly stationary set is stationary. We show that if $0^\sharp$ does not exist then every weakly stationary set is stationary. Also there is a weakly stationary nonstationary set on $[\omega_2]^{\aleph_1}$ if and only if Chang's conjecture holds. From a ω_1 -Erdős cardinal, we get a model in which $2^{\aleph_0} > \omega_2$ and all the subsets of $2^{\aleph_0}$ of order type ω_1 form a weakly stationary set which is nonstationary.

In this paper, we consider the problem of generalizing a theorem of Kueker's to uncountable. It is shown that the possibility of certain proper generalization depends on set theory and any negative statement on this matter is a large cardinal property.

We use standard set-theoretic terminology. All the undefined terms are referred to Jech's book [8].

1. THE QUESTION

Let μ be a regular cardinal. For any set A of cardinality at least μ^+ , let $[A]^\mu = \{x \subseteq A : |x| = \mu\}$ and $[A]^{<\mu} = \{x \subseteq A : |x| < \mu\}$, where $|x|$ is the cardinality of x . In particular, $[A]^{<\omega}$ is the set of all finite subsets of A .

Let A be a set of cardinality at least μ^+ . For any subset C of $[A]^\mu$, C is called a *club* in $[A]^\mu$ (*closed* and *unbounded*) if for every $x \in [A]^\mu$ there is a $y \in C$ so that $x \subseteq y$, and for any sequence $\langle x_\alpha : \alpha < \beta \rangle$ from C of length $\beta \leq \mu$ we have $\cup\{x_\alpha : \alpha < \beta\} \in C$ whenever for all $\alpha < \eta < \beta$, $x_\alpha \subseteq x_\eta$. For any subset S of $[A]^\mu$, S is *stationary* in $[A]^\mu$ if for every club $C \subseteq [A]^\mu$ $S \cap C$ is not empty. Let

$$C_\mu(A) = \{X \subseteq [A]^\mu : \text{for some club } C \subseteq [A]^\mu \ X \supseteq C\}.$$

Then $C_\mu(A)$ is a μ^+ -complete normal filter, i.e., $C_\mu(A)$ is closed under intersection of μ many elements and under diagonal intersection of its members.

The following theorem is due to Keuker [7], which gives a very nice characteristic of clubs of $[A]^\omega$.

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Theorem. Let X be a subset of $[A]^\omega$. Then the following are equivalent:

- (1) $X \in \mathbb{C}_\omega(A)$;
- (2) there is a function $f: [A]^{<\omega} \rightarrow A$ such that for any $x \in [A]^\omega$ if for all $e \in [x]^{<\omega}$ $f(e) \in x$, then $x \in X$;
- (3) there is a function $f: [A]^{<\omega} \rightarrow [A]^\omega$ such that for any x in $[A]^\omega$ if for all $e \in [x]^{<\omega}$ $f(e) \subseteq x$, then $x \in X$.

In general, (3) can still be used to define $\mathbb{C}_\mu(A)$, i.e., for $X \subseteq [A]^\mu$ $X \in \mathbb{C}_\mu(A)$ if and only if there is a function $f: [A]^{<\omega} \rightarrow [A]^\mu$ so that for any $x \in [A]^\mu$ if for all $e \in [x]^{<\omega}$ $f(e) \subseteq x$, then $x \in X$.

Let $f: [A]^{<\omega} \rightarrow A$ be a function. Let $C(f, \mu) = \{x \in [A]^\mu: f''[x]^{<\omega} \subseteq x\}$ where $f''[x]^{<\omega} = \{f(e): e \in [x]^{<\omega}\}$. It is easy to see that $C(f, \mu)$ is a club in $[A]^\mu$. Now the question is whether for $\mu > \omega$ it is still the case that for every club $C \subseteq [A]^\mu$ there is an $f: [A]^{<\omega} \rightarrow A$ so that $C(f, \mu) \subseteq C$. We will see that this question cannot be answered in ZFC, the Zermelo–Fraenkel set theory plus the axiom of choice.

Let us call a subset C of $[A]^\mu$ a *strong club* if there is a function f from $[A]^{<\omega}$ to A so that $C = C(f, \mu)$. For $S \subseteq [A]^\mu$, S is *weakly stationary* if for every strong club $C \subseteq [A]^\mu$ $S \cap C$ is not empty. That is, S is weakly stationary if and only if for every function $f: [A]^{<\omega} \rightarrow A$ there is an $x \in S$ so that x is closed under f , i.e., $f''[x]^{<\omega} \subseteq x$. Since every strong club is a club, every stationary set is weakly stationary. By Kueker's theorem, for $\mu = \omega$, every weakly stationary set is stationary. The question above can now be stated as whether there is a weakly stationary set on $[A]^\mu$ which is not stationary for $\mu > \omega$.

Given A with $|A| > \mu$, let $\mathbb{C}_\mu^s(A) = \{X \subseteq [A]^\mu: \exists \text{ a strong club } C \subseteq X\}$. Then $\mathbb{C}_\mu^s(A)$ is a σ -complete normal filter (see [5] for a proof). By Kueker's theorem, $\mathbb{C}_\omega(A) = \mathbb{C}_\omega^s(A)$. Is $\mathbb{C}_\mu(A) = \mathbb{C}_\mu^s(A)$ for $\mu > \omega$?

2. WHEN NO $0^\sharp$

In this section we show that if $0^\sharp$ does not exist then $\mathbb{C}_\mu(A) = \mathbb{C}_\mu^s(A)$.

First let us see which sets could be counterexamples.

Proposition 1. Let $\kappa \geq \mu^+ > \mu > \omega$. If $S \subseteq [\kappa]^\mu$ is weakly stationary but not stationary, then

$$S \cap \{x \in [\kappa]^\mu: x \cap \mu \neq \mu\}$$

is weakly stationary.

Proof. We need only consider the case $\mu = \aleph_1$. The general case follows by the same proof.

Without loss of generality, let $F: [\kappa]^{<\omega} \rightarrow [\kappa]^{\aleph_1}$ be such that for all $x \in [\kappa]^{\aleph_1}$ if $F(e) \subseteq x$ for all $e \in [x]^{<\omega}$, then $x \notin S$.

For $\alpha < \omega_1$ define $f_\alpha: [\kappa]^{<\omega} \rightarrow \kappa$ so that for all $e \in [\kappa]^{<\omega}$

$$F(e) = \{f_\alpha(e): \alpha < \omega_1\}.$$

Let $C_\alpha = \{x \subseteq \kappa : |x| = \aleph_1 \text{ \& } \forall e \in [x]^{<\omega} f_\alpha(e) \in x\}$ for $\alpha < \omega_1$. Let for $\alpha \geq \omega_1$ $C_\alpha = [\kappa]^{\aleph_1}$. Let $C = \{x \subseteq \kappa : |x| = \aleph_1 \text{ \& } \forall \alpha \in x \ x \in C_\alpha\}$. Then by normality, C contains a strong club. Hence $S \cap C$ is weakly stationary. Notice that $x \in S \cap C$ if and only if $x \in S$ and (either $x \cap \omega_1 = 0$ or $\forall \alpha \in x \cap \omega_1$ $x \in C_\alpha$). Since $\{x \subseteq \kappa : |x| = \aleph_1 \text{ \& } x \cap \omega_1 = 0\}$ is not weakly stationary, $T = \{x \in S : |x| = \aleph_1 \text{ \& } x \cap \omega_1 \neq 0 \text{ \& } \forall \alpha \in x \cap \omega_1 \ x \in C_\alpha\}$ is weakly stationary. For $x \in T$, $\forall e \in [x]^{<\omega} \forall \alpha \in x \cap \omega_1 \ f_\alpha(e) \in x$. If $x \cap \omega_1 = \omega_1$, then $\forall e \in [x]^{<\omega} F(e) \subseteq x$. This is impossible. Hence $\forall x \in T \ x \cap \omega_1 \neq \omega_1$. ■

Since $\{x \subseteq \kappa : |x| = \mu \text{ \& } x \supseteq \mu\}$ is a club, the proposition tells us that if there is a weakly stationary nonstationary set then there is a canonical one.

The following lemma is from [5] (cf. Lemma 1.2 (5)).

Lemma 2. *Let $\omega < \mu < \mu^+ \leq \kappa$ be cardinals. Then there is a function $f : [\kappa]^{<\omega} \rightarrow \kappa$ so that $\forall x \in [\kappa]^\mu \ x \supseteq \mu \text{ \& } f''[x]^{<\omega} \subseteq x \Rightarrow x \cap \mu^+ \in \mu^+$. Moreover if $\mu = \aleph_1$, then $\forall x \in [\kappa]^{<\omega} \ x \cap \omega_1 \neq \omega_1 \text{ \& } f''[x]^{<\omega} \subseteq x \Rightarrow x \cap \omega_1 \in \omega_1$ and the order type of $x \cap \omega_2 \leq \omega_1$.*

By use D_f to denote this particular strong club without mentioning μ and κ from now on. Immediately we have for all $x \in D_f \ x \cap \omega_1 \in \omega_1 \Leftrightarrow x \cap \omega_1 \neq \omega_1$ if $\mu = \omega_1$.

Corollary 3. *For μ and κ as in the lemma, the following are equivalent:*

- (1) $C \in \mathbb{C}_\mu(\kappa) \Leftrightarrow \exists D \in \mathbb{C}_\mu^s(\kappa) \ \forall x \in D \ x \cap \mu^+ \in \mu^+ \Rightarrow x \in C$;
- (2) S is stationary in $[\kappa]^\mu \Leftrightarrow$ for each strong club D there is an x in $S \cap D$ so that $x \cap \mu^+ \in \mu^+$;
- (3) If $S \subseteq [\kappa]^\mu$ is weakly stationary but not stationary, then

$$S \cap \{x \in [\kappa]^\mu : x \cap \mu \neq \mu\}$$

is weakly stationary.

Proof. We need only to show (1) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (3). Assume that $S \subseteq [\kappa]^\mu$ is weakly stationary but not stationary. Let C be a club disjoint from S . Let D be a strong club so that $\forall x \in D \ x \cap \mu^+$ is an ordinal implying $x \in C$. Then $S_0 = S \cap D \cap D_f$ is weakly stationary and $S_0 \subseteq \{x \in [\kappa]^\mu : x \cap \mu \neq \mu\}$.

(3) $\Rightarrow$ (1). Let $C \in \mathbb{C}_\mu(\kappa)$. Without loss of generality, we may assume that C is a club. Let $S = [\kappa]^\mu - C$. Then S is not stationary. By (3),

$$S \cap \{x \in [\kappa]^\mu : \mu \subseteq x\}$$

is not weakly stationary. Let D be a strong club disjoint from it and be such that $\forall x \in D \ \mu \in x$. Then $\forall x \in D \ x \cap \mu^+ \in \mu^+ \Rightarrow x \in C$. Notice that the other direction of (1) follows from the fact that $\{x \in [\kappa]^\mu : x \cap \mu^+ \in \mu^+\}$ is a club.

This finishes the proof. ■

From Proposition 1 and Corollary 3 we can conclude that the club filter $\mathbb{C}_\mu(\kappa)$ is generated by the strong club filter and the set $\{x \in [\kappa]^\mu : x \cap \mu^+ \in \mu^+\}$, which is well known [2, 6] but our proof is simpler.

Theorem 4. *Let $\omega < \mu < \kappa$ be cardinals. Then the following are equivalent:*

- (1) $\mathbb{C}_\mu(\kappa) \neq \mathbb{C}_\mu^s(\kappa)$;
- (2) $\{x \in [\kappa]^\mu : x \cap \mu \neq \mu\}$ is weakly stationary;
- (3) $\exists \lambda \leq \mu$ $\lambda > \omega$ is regular and $\{x \in [\kappa]^\mu : x \cap \lambda \in \lambda\}$ is weakly stationary.

Proof. We need only to prove (2) $\Rightarrow$ (3).

Let $T = \{x \in [\kappa]^\mu : x \cap \mu \neq \mu \text{ \& } \mu \in x\}$. By (2), T is weakly stationary. For $x \in T$ let $f(x) =$ the least $\alpha \in x$ so that $\exists \beta < \alpha$ $\beta \notin x$. By normality there must be $\beta_0 \leq \mu$ and $S \subseteq T$ weakly stationary so that $\forall x \in S$ $f(x) = \beta_0$. Take λ to be the least such ordinal. Let $S \subseteq T$ be weakly stationary such that $\forall x \in S$ $f(x) = \lambda$. Since the strong club filter is countably complete, $\lambda \geq \omega_1$.

Claim. $\lambda = cf\lambda$.

Otherwise, let $\beta = cf\lambda < \lambda$. Let $g: \beta \rightarrow \lambda$ be a cofinal sequence. By the choice of λ , $\{x \in [\kappa]^\mu : \beta \in x \text{ \& } x \cap \beta \neq \beta\}$ is not weakly stationary. Let C be a strong club in $[\kappa]^\mu$ such that $\forall x \in C$ $\beta \in x$ and $\beta \subseteq x$ and $g''x \subseteq x$. Then $C \cap S$ is weakly stationary. For $x \in C \cap S$, $\exists \eta < \lambda$ $\eta \notin x$, hence for some $\alpha < \beta$ $\exists \eta < g(\alpha)$ $\eta \notin x$. Let $h(x)$ be such $g(\alpha)$. By normality, we get some λ' smaller than λ and some weakly stationary $S' \subseteq S \cap C$ so that for all $x \in S'$ $\lambda' \in x$ and $\exists \eta < \lambda'$ $\eta \notin x$. This contradicts to the choice of λ .

This finishes the proof. ■

Remark. The theorem is also true for $\kappa = \mu$. In this case, (2) simply says that κ is Jonsson, i.e., every structure (for a countable language) of cardinality κ has a proper elementary substructure of the same cardinality. Notice that for $X \subseteq [\kappa]^\kappa$, $X \in \mathbb{C}_\kappa(\kappa) \Leftrightarrow \kappa \in X$. Hence $\mathbb{C}_\kappa(\kappa) \neq \mathbb{C}_\kappa^s(\kappa)$ if and only if $\{\kappa\} \notin \mathbb{C}_\kappa(\kappa)$ if and only if $\{x \in [\kappa]^\kappa : x \neq \kappa\}$ is weakly stationary. Now the rest follows from the same proof as in the case $\mu < \kappa$.

Corollary 5. *The following are equivalent:*

- (1) *There is a weakly stationary nonstationary subset of $[\omega_2]^{\aleph_1}$.*
- (2) *Chang's conjecture holds, i.e., for any model $\mathfrak{A} = \langle A, U \rangle$ of countable language with one unary predicate U , if $\omega_2 \subseteq A$ and $|A| = \aleph_2$ and $|U| = \aleph_1$, then $\mathfrak{A}$ has an elementary submodel $\mathfrak{B} = \langle B, V \rangle$ so that $V = U \cap B$ and $|B| = \aleph_1$ and $|V| = \aleph_0$.*

Proof. It is well known that Chang's conjecture holds if and only if for every $f: [\omega_2]^{<\omega} \rightarrow \omega_2$ there is an $x \in [\omega_2]^{\aleph_1}$ so that $|x \cap \omega_1| \leq \aleph_0$ and $\forall e \in [x]^{<\omega}$ $f(e) \in x$. ■

More generally, we have the following, as the referee pointed out.

Corollary 5'. *For $\omega < \mu \leq \kappa$, the following are equivalent:*

- (1) *There is a weakly stationary nonstationary subset of $[\kappa]^\mu$.*

(2) *There is a regular cardinal $\lambda \leq \mu$ so that $(\kappa, \lambda) \rightarrow (\mu, < \lambda)$, where $(\kappa, \lambda) \rightarrow (\mu, < \lambda)$ means that for any model $\mathcal{A} = \langle A, U \rangle$ of countable language with one unary predicate U , if $\kappa \subseteq A$, $|A| = \kappa$ and $|U| = \lambda$, then $\mathcal{A}$ has an elementary submodel $\mathcal{B} = \langle B, V \rangle$ so that $V = U \cap B$, $|B| = \mu$ and $|V| < \lambda$.*

Remark. A special case of this corollary says that κ is Jonsson if and only if for some regular $\lambda < \kappa$, $(\kappa, \lambda) \rightarrow (\kappa, < \lambda)$. Hence we have a new proof of this known result, also as the referee pointed out.

Proposition. $\square_{\omega_1}$ implies that there is no weakly stationary nonstationary subset of $[\omega_2]^{\aleph_1}$.

It is known that $\square_{\omega_1}$ implies the negation of Chang's conjecture. According to the referee, "no explicit proof seems to be available." We give a direct proof of this below.

$\square_{\omega_1}$ is the following combinatorial principle, due to Jensen [9].

There is a sequence $\langle C_\alpha : \alpha \in \omega_2 \text{ \& } \alpha \text{ is a limit ordinal} \rangle$ so that

- (1) for each α $C_\alpha \subseteq \alpha$ is a club in α ;
- (2) for each α if $cf \ \alpha = \omega$ then the order type of $C_\alpha < \omega_1$;
- (3) if $\alpha < \beta$ and α is a limit point of C_β , then $C_\alpha = \alpha \cap C_\beta$.

Proof of the proposition. Let lim be the set of limit ordinals below ω_2 . let $\langle C_\alpha : \alpha \in \text{lim} \rangle$ be a $\square$ -sequence. For each C_α let C'_α be the set of limit points of C_α . We define a function $f: [\omega_2]^{<\omega} \rightarrow [\omega_2]^{\leq \omega}$ as follows.

$$f(\{\alpha\}) = \begin{cases} C_\alpha \cup ot(C_\alpha) + 1, & \text{if } cf \ \alpha = \omega; \\ \{\alpha + 1, \text{ the largest limit ordinal } \leq \alpha\}, & \text{otherwise.} \end{cases}$$

$$f(\{\alpha, \beta\}) = \begin{cases} \{\text{the least } \eta \in C'_\beta \text{ so that } \eta > \alpha\}, & \text{if } cf \ \beta = \omega_1; \\ \{\text{the least } \eta > \beta \text{ so that } cf \ \eta = \omega\}, & \text{otherwise,} \end{cases}$$

elsewhere f is defined arbitrarily, where $ot(x)$ is the order type of x .

Claim. There is no $X \in [\omega_2]^{\aleph_1}$ so that $ot(X) = \omega_1$ and $X \cap \omega_1 \in \omega_1$ and for all $e \in [X]^{<\omega}$ $f(e) \subseteq X$.

Assume otherwise. Let X be a counterexample. Let $\delta = \sup(X)$. Then $cf \ \delta$ is ω_1 . Let $\gamma = X \cap \omega_1$. Now $D = \{\sup(X \cap \beta) : \beta \in X\}$ is a club in δ . Since C_δ is a club in δ of order type ω_1 , C'_δ is a club in δ of order type ω_1 . We can pick some $\eta \in D \cap C'_\delta$ so that $ot(C_\eta) \gg \gamma$, since $C_\eta = \eta \cap C_\delta$ for any $\eta \in C'_\delta$. Since $\eta \in D$ is a limit ordinal, there is a limit ordinal $\beta \in X$ so that $\eta = \sup(X \cap \beta)$ (notice that X is closed under predecessors). Fix such a β . If $cf \ \beta = \omega$, then $\eta = \beta$ for $C_\beta \subseteq X$. But then $ot(C_\beta) \in X$. This is impossible. Hence $cf \ \beta = \omega_1$. Now $\eta < \beta$ and $\eta \in C'_\beta$ and $\eta = \sup(\eta \cap C'_\beta)$. Therefore, $C_\eta = \eta \cap C_\beta$, and for every $\xi \in C'_\beta \cap X \cap \eta$, $\xi \in C'_\eta \cap X$ and $C_\xi = \xi \cap C_\eta$. One can then find a $\xi \in C'_\eta \cap X$ so that $ot(C_\xi) > \gamma$. We get a contradiction. This finishes the proof of the claim, hence the proposition. ■

The previous corollary says that $\mathfrak{C}_{\omega_1}(\omega_2) \neq \mathfrak{C}_{\omega_1}^s(\omega_2)$ is really a large cardinal property. Next we want to show the same is true in general.

Theorem 6. *If for some $\omega < \mu \leq \kappa$ cardinals $C_\mu(\kappa) \neq C_\mu^s(\kappa)$, then $\forall a \subseteq \omega$ $a^\sharp$ exists.*

Proof. Let $\mu < \kappa$ be regular uncountable cardinals. Assume that the club filter contains the strong club filter properly on $[\kappa]^\mu$. Let $\lambda \leq \mu$ be a regular uncountable cardinal so that $\{x \in [\kappa]^\mu : x \cap \lambda \in \lambda\}$ is weakly stationary. Given $a \subseteq \omega$, let $f: [L_\kappa[a]]^{<\omega} \rightarrow L_\kappa[a]$ be coding skolem functions for $L_\kappa[a]$. Let $X \subseteq L_\kappa[a]$ be closed under f and $|X| = \mu$ and $X \cap \lambda$ is an ordinal smaller than λ .

Let $\pi: X \approx L_\alpha[a]$ be the transitive collapse of X . We have

$$\pi^{-1}: L_\alpha[a] \rightarrow L_\kappa[a]$$

is an elementary embedding and $|\alpha| = \mu$, $\pi^{-1}(X \cap \lambda) = \lambda$.

Let $\gamma = X \cap \lambda$. Define U as follows:

$$\text{for } x \in L[a] \cap P(\gamma), \quad x \in U \Leftrightarrow \gamma \in \pi^{-1}(x).$$

Then $L[a]^\gamma/U$ is well founded. Therefore, we get an elementary embedding

$$j: L[a] \rightarrow L[a] \approx L[a]^\gamma/U$$

which is nontrivial. By Kunen's work, $a^\sharp$ exists. ■

From this theorem we see that if $0^\sharp$ does not exist then Kueker's theorem generalizes to uncountable. In particular, if $V = L$, then $C_\mu = C_\mu^s$.

3. TO GET $C_{\omega_1}(2^{\aleph_0}) \neq C_{\omega_1}^s(2^{\aleph_0})$ WITH $2^{\aleph_0} > \omega_2$

In this section we show how to get a weakly stationary nonstationary subset of $[2^{\aleph_0}]^{\aleph_1}$.

Recall that (see [1, 3, 4]) for $\omega \leq \alpha \leq \kappa$, κ is α -Erdős if and only if whenever $C \subseteq \kappa$ is a club and $f: [C]^{<\omega} \rightarrow \kappa$ is regressive (i.e., $f(e) < \text{the least member of } e$) there is a $D \subseteq C$ such that $\forall n \in \omega$ $|f''[D]^n| = 1$ and the order type of D is α .

A useful fact of α -Erdős cardinals is the following (see [4] for a proof).

Fact. Let $\omega \leq \alpha$ be a limit ordinal. Let $\kappa \geq \alpha$. If κ is a α -Erdős, then κ is inaccessible, and for every structure $\mathfrak{A} = \langle V_\kappa, \in, \dots \rangle$ of a countable language, there is a set $H \subseteq \kappa$ of indiscernibles (for $\mathfrak{A}$) of order type α satisfying the following property:

- (*) for every $\gamma \in H$, $\mathfrak{A}|_\gamma = \langle V_\gamma, \in, \dots \rangle$ is an elementary submodel of $\mathfrak{A}$ and $H - \gamma$ is a set of indiscernibles for $\langle \mathfrak{A}, \xi \rangle_{\xi < \gamma}$.

(For a pure combinatorial statement of this fact, see [3].)

Theorem 7. *If κ is ω_1 -Erdős, then*

$$(\kappa)^{\omega_1} = \{x \in [\kappa]^{\aleph_1} : \text{the order type of } x \text{ is } \omega_1\}$$

is weakly stationary.

Proof. Let $f: [\kappa]^{<\omega} \rightarrow \kappa$. We need to find an x in $(\kappa)^{\omega_1}$ so that x is closed under f .

Let $R = \{ \langle n, \alpha_1, \dots, \alpha_n, \beta \rangle : f(\{\alpha_1, \dots, \alpha_n\}) = \beta \}$. Consider the structure $\mathfrak{A} = \langle V_\kappa, \in, R, <, \dots \rangle$, where $<$ is a well ordering of V_κ and ... represent a complete set of skolem functions.

Take $H \in (\kappa)^{\omega_1}$ to be a set of indiscernibles for $\mathfrak{A}$ satisfying (*). Let $\mathfrak{B}$ be the skolem hull of H in $\mathfrak{A}$. Then $|\mathfrak{B}| = \aleph_1$ and $\mathfrak{B} \cap \kappa$ has cardinality $\aleph_1$.

Enumerating H in increasing order, $H = \{c_\alpha : \alpha < \omega_1\}$.

Claim. (1) $\forall \alpha < \omega_1$ $\mathfrak{B} \cap c_\alpha$ is countable;

(2) $\mathfrak{B} \cap \kappa$ has order type ω_1 .

Assume the claim, let $x = \mathfrak{B} \cap \kappa$, then $f''[x]^{<\omega} \subseteq x$.

For (1), let τ be a term in $n + m + 1$ free variables. For $d \in {}^m H$, $x, y \in {}^n H$, $d < c_\alpha < x < y$, assuming that they are in the same order, if $\tau(d, c_\alpha, x) < c_\alpha$, then $\tau(d, c_\alpha, y) < c_\alpha$ by indiscernibility. In case $\tau(d, c_\alpha, x) < c_\alpha$, we must have $\tau(d, c_\alpha, x) = \tau(d, c_\alpha, y)$. For let $\gamma = \tau(d, c_\alpha, x)$, by (*), $H - c_\alpha$ is a set of indiscernibles for the structure $\langle \mathfrak{A}, \xi \rangle_{\xi < c_\alpha}$. Then $\gamma = \tau(d, c_\alpha, y)$. Now $\mathfrak{B} \cap c_\alpha$ is completely determined by the next ω indiscernibles and $H \cap c_\alpha$. Since there are only countably many terms, $\mathfrak{B} \cap c_\alpha$ is countable.

For (2), let τ be a term of n free variables. Let $x \in {}^n H$. Assume that $\tau(x) \in \kappa$. Let $c \in H$ be such that $x < c$. By (*), $\mathfrak{A} \upharpoonright c$ is an elementary submodel of $\mathfrak{A}$. Since τ is a skolem term, we must have $\tau(x) < c$. Therefore,

$$\mathfrak{B} \cap \kappa = \bigcup \{ \mathfrak{B} \cap c_\alpha : \alpha < \omega_1 \}.$$

It follows that $\mathfrak{B} \cap \kappa$ has order type ω_1 . ■

Remarks. By a similar argument one can show that if κ is μ -Erdős, then $\{x \in [\kappa]^\mu : ot(x) = \mu \text{ \& } \mu \neq \kappa\}$ is weakly stationary, hence $C_\mu(\kappa) \neq C_\mu^s(\kappa)$. At the first time, we used Ramsey cardinal. The referee suggested that one could use ω_1 -Erdős cardinal instead.

Lemma 8. Let $\mathbf{P}$ be a c.c.c. forcing. Let $\kappa \geq \omega_2$ be a cardinal. Let $\mu \leq \kappa$ be a cardinal. Then for any weakly stationary $S \subseteq [\kappa]^\mu$, in $V^\mathbf{P}$, S is still weakly stationary.

Proof. Let $p \Vdash f: [\kappa]^{<\omega} \rightarrow \kappa$. Let $S \subseteq [\kappa]^\mu$ be weakly stationary.

For $e \in [\kappa]^{<\omega}$, let $g(e) = \{\alpha < \kappa : \exists q \leq p \ q \Vdash f(e) = \alpha\}$. By c.c.c., $g(e)$ has cardinality at most countable. Hence $g: [\kappa]^{<\omega} \rightarrow [\kappa]^\omega$. Let

$$D = \{x \in [\kappa]^\mu : \forall e \in [x]^{<\omega} g(e) \subseteq x\}.$$

Then D contains a strong club in $[\kappa]^\mu$. Let $X \in S \cap D$. Then

$$\forall e \in [X]^{<\omega} p \Vdash f(e) \in X$$

since $\forall e \in [X]^{<\omega} p \Vdash f(e) \in g(e)$.

Therefore, in $V^\mathbf{P}$, S remains weakly stationary. ■

Theorem 9. *If it is consistent that there is a ω_1 -Erdős cardinal with ZFC then so is ZFC plus $2^{\aleph_0} > \omega_2$ and $\{x \subseteq 2^{\aleph_0} : \text{the order type of } x \text{ is } \omega_1\}$ is weakly stationary in $[2^{\aleph_0}]^{\aleph_1}$.*

Proof. Let κ be ω_1 -Erdős. Let $\mathbf{P}$ be the forcing notion adding κ many Cohen reals. Then $\mathbf{P}$ has the c.c.c. and $\|\bar{\mathbf{P}}\| 2^{\aleph_0} = \kappa$.

Now apply Theorem 7 and Lemma 8 to finish the proof. ■

Remarks. (1) One can make $2^{\aleph_0}$ to be anything it could so that

$$\{x \subseteq 2^{\aleph_0} : |x| = \aleph_1 \text{ \& } |x \cap \omega_1| \leq \aleph_0\}$$

is weakly stationary. To see this, let κ be ω_1 -Erdős. One first Levy collapses κ to ω_2 via countable conditions. One then adds Cohen reals. The result follows from that by Baumgartner [3] Chang's conjecture holds in the first extention and then apply Lemma 8 inside that model.

(2) Notice that, as the referee pointed out, the assumption of existence of ω_1 -Erdős cardinal is optimal. Using the methods of [4] (cf. Theorem 1.7, p. 65), one can show that if for some regular $\lambda \leq \mu$, $(\kappa, \lambda) \rightarrow (\mu, < \lambda)$, which follows from $C_\mu(\kappa) \neq C_\mu^s(\kappa)$, then for some $\xi \leq \kappa$, ξ is μ -Erdős in the Dodd-Jensen core model K .

(3) By a suggestion of the referee, we now show how to apply a method of Silver to make $2^\omega = \aleph_3$ and $\{x \in [\omega_3]^{\aleph_1} : \text{ot}(x) = \omega_1\}$ is weakly stationary.

We start with a model $\mathfrak{M}$ of ZFC in which MA_{ω_1} holds and κ is ω_1 -Erdős. We now work inside $\mathfrak{M}$. Let $\mathbf{P} = \text{Col}(\omega_3, \kappa)$ be the Levy collapsing by $\aleph_1$ -closed conditions, i.e., $\mathbf{P}$ is the set of partial functions p satisfying

- (1) $\text{dom}(p) \subseteq \omega_2 \times \kappa$ and $|\text{dom}(p)| < \aleph_2$;
- (2) for all $\langle \alpha, \beta \rangle \in \text{dom}(p)$ $p(\alpha, \beta) \in \beta$.

For two conditions p and q , $q \leq p$ if and only if $q \supseteq p$.

It is standard that $\mathbf{P}$ is $\aleph_1$ -closed and $\mathbf{P}$ has the κ -c.c. and if G is a $\mathbf{P}$ -generic over $\mathfrak{M}$ then $\mathfrak{M}[G] \models \kappa = \omega_3$. Now we want to show that in $\mathfrak{M}[G]$ the set $\{x \in [\omega_3]^{\aleph_1} : \text{ot}(x) = \omega_1\}$ is weakly stationary. Then one simply adds $\aleph_3$ many Cohen reals to $\mathfrak{M}[G]$.

Let us go back to $\mathfrak{M}$.

Let $p \Vdash f : [\kappa]^{<\omega} \rightarrow \kappa$. We need some $q \leq p$ and some X so that

$$q \Vdash \text{ot}(X) = \omega_1 \text{ \& } \forall e \in [X]^{<\omega} f(e) \in X.$$

Notice that since κ is inaccessible, $\mathbf{P} \subseteq V_\kappa$. Define $R \subseteq [\kappa]^{<\omega} \times \kappa \times \mathbf{P}$ by

$$\langle e, \alpha, r \rangle \in R \Leftrightarrow r \leq p \text{ \& } r \Vdash f(e) = \alpha.$$

Consider the structure $\mathfrak{A} = \langle V_\kappa, \in, \omega_1, \omega_2, \mathbf{P}, \{p\}, R, \Delta, \dots \rangle$, where Δ is a well ordering of the universe of $\mathfrak{A}$, and $\dots$ represents a complete set of skolem functions of $\mathfrak{A}$. Being ω_1 -Erdős, let $H \subseteq \kappa$ be a set of indiscernibles for $\mathfrak{A}$ satisfying (*) of order type ω_1 . Let $\mathfrak{B}$ be the skolem hull of H in $\mathfrak{A}$. By the

proof of Theorem 7, we have that $|\mathfrak{B}| = \aleph_1$ and $\mathfrak{B} \cap \kappa$ has order type ω_1 . In particular, for each $\beta \in \mathfrak{B} \cap \kappa$, $\mathfrak{B} \cap \beta$ is countable.

Claim. $\mathbf{P} \cap \mathfrak{B}$ has the countable chain condition.

First, let τ be a term of n free variables. Let $x < y$ be in ${}^n H$. Assume that $\mathfrak{B} \Vdash \tau(x)$ is a condition. Then $\mathfrak{B} \Vdash \tau(x)$ and $\tau(y)$ are compatible.

Assume otherwise. Then $\mathfrak{B} \Vdash \tau(x)$ and $\tau(y)$ are incompatible. That is, $\mathfrak{B} \Vdash \exists \alpha, \beta \langle \alpha, \beta \rangle \in \text{dom}(\tau(x)) \cap \text{dom}(\tau(y))$ and $\tau(x)(\alpha, \beta) \neq \tau(y)(\alpha, \beta)$. The least such pair $\langle \alpha, \beta \rangle$ is definable from x, y via τ . By indiscernibility, for this pair $\langle \alpha, \beta \rangle$, for $x' < y'$ in ${}^n H$, $\tau(x')(\alpha, \beta) \neq \tau(y')(\alpha, \beta)$. There are $\aleph_1$ many such sequences, but $\mathfrak{B} \cap \beta$ is only countable.

Therefore, $\beta \Vdash \tau(x)$ and $\tau(y)$ are compatible.

Then it is clear that $\mathbf{P} \cap \mathfrak{B}$ has the countable chain condition.

For $e \in [\mathfrak{B} \cap \kappa]^{<\omega}$, let $D_e = \{r \in \mathfrak{B} \cap \mathbf{P} : r \leq p \text{ \& } \exists \alpha \ r \Vdash f(e) = \alpha\}$. Then each D_e is dense below p in $\mathfrak{B} \cap \mathbf{P}$. Now applying MA_{ω_1} to these D_e 's, we get a filter $G \subseteq \mathfrak{B} \cap \mathbf{P}$ meeting all D_e for $e \in [\mathfrak{B} \cap \kappa]^{<\omega}$. Since $\mathbf{P}$ is $\aleph_1$ -closed, $\cup G$ is a condition. Let $q = \cup G$ and $X = \mathfrak{B} \cap \kappa$. Then $q \Vdash -ot(X) = \omega_1$ & $\forall e \in [X]^{<\omega} f(e) \in X$. Notice that $p \in G$ since $p \in \mathfrak{B}$ and G is a filter. Hence $q \leq p$.

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