EUCLIDEAN METRIC AND FLAT METRIC OUTSIDE A COMPACT SET

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ABSTRACT. We investigate to what extent there is the gap between Euclidean metrics and flat metrics outside a compact set in complete Riemannian planes.

0. INTRODUCTION

Let g be a complete Riemannian metric on $\mathbf{R}^n$ and let g_0 be the Euclidean metric. We say that $(\mathbf{R}^n, g)$ is *Euclidean outside a compact set* if $g - g_0$ has compact support. L. Green and R. Gulliver [4] have proved that a plane which is Euclidean outside a compact set is Euclidean if there is no conjugate point. In the present note we discuss the theorem from the viewpoint of the gap-phenomenon shown by R. Greene and H. Wu [5]. They have shown in the proof of Theorem 1 of [5] that $(\mathbf{R}^n, g)$, $n \geq 3$, is Euclidean outside a compact set if it is flat outside some compact set. Namely, the metric g is a modification of Euclidean metric just on a compact set. This phenomenon is not true without additional condition if $n = 2$. We prove the following.

Theorem. *Let M be a Riemannian plane which is flat outside a compact set K . If the total curvature of M is zero and all geodesics in $M - K$ are minimizing, then M is Euclidean outside some compact set.*

Since all geodesics are minimizing in a simply connected Riemannian manifold without conjugate points, we have the following from the theorem of L. Green and R. Gulliver.

Corollary. *Let M be a Riemannian plane without conjugate points. If the Gaussian curvature is zero outside a compact set and the total curvature of M is zero, then M is Euclidean.*

The higher dimensional case for Corollary was treated in [7] by a different method. We first give examples which show that the assumption of the Theorem is necessary. Next we prove the Theorem.

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1. Example. (A) (See [4], p. 44). Let $g = dr^2 + F(r)^2 d\theta^2$ in polar coordinates, where $F(r) = 2r - (1 + \exp(2r/(r-1)))/2$ for $0 \leq r \leq 1$ and $F(r) = 2r - 1/2$ for $r \geq 1$. The Gaussian curvature of $(\mathbf{R}^2, g)$ is nonpositive and flat on $r \geq 1$. This implies that the Theorem is not true without the assumption on the total curvature.

(B) The idea is due to Y. Nasu [8, p. 79, see Figure 2, p. 80]. Let f be a C^∞ nonnegative function of compact connected support on $\mathbf{R}$. Let $S_1 = \{(x, y, z) \in \mathbf{R}^3; -\infty < x < \infty, y \leq 0, z = 0\}$, $S_2 = \{(x, y, z) \in \mathbf{R}^3; 0 \leq z \leq f(x), x \in \text{Supp}(f), y = 0\}$, and $S_3 = \{(x, y, z) \in \mathbf{R}^3; -\infty < x < \infty, y \geq 0, z = f(x)\}$. The surface $S' = S_1 \cup S_2 \cup S_3$ is smooth except on the curve bounding S_2 . Smoothing out S' we have a smooth surface S such that the Gaussian curvature is zero outside a compact set and the total curvature is zero. This implies that the Theorem is false without the assumption of minimality of geodesics in $M - K$.

(C) A complete Riemannian manifold $M = (\mathbf{R}^2, g)$ of nonnegative (or nonpositive) Gaussian curvature is clearly Euclidean if the total curvature is zero. The same phenomenon is not true in the case "outside a compact set". Namely, there is a Riemannian metric g on $\mathbf{R}^2$ such that the total curvature is zero and the Gaussian curvature is nonnegative (or nonpositive) outside a compact set (not identically zero): Let f be a function on $[0, \infty)$ such that

- (1) $f(r) > 0$ on $[0, \infty)$ and $f(r) = 1$ on $[0, a]$ for some $a > 0$.
- (2) $f(r) \geq 1$ (or $f(r) \leq 1$) on $[b, \infty)$ for some $b > a$ and $f(r) \rightarrow 1$ as $r \rightarrow \infty$.
- (3) $f'(r) \leq 0$ (or $f'(r) \geq 0$) on $[b, \infty)$, but not identically zero. Put

$$F(r) = \int_0^r f(r) dr.$$

We consider a Riemannian metric on $\mathbf{R}^2$ given by

$$g = dr^2 + F(r)^2 d\theta^2$$

in polar coordinates. Then, if $G(r, \theta)$ is the Gaussian curvature at (r, θ) , then

$$G(r, \theta) = -\frac{F''(r)}{F(r)} = -\frac{f'(r)}{F(r)} \geq 0 \quad (\text{or } \leq 0)$$

for $r \geq b$. Thus, $M = (\mathbf{R}^2, g)$ has nonnegative (or nonpositive, resp.) Gaussian curvature outside a compact set. We calculate the total curvature as follows.

$$\begin{aligned} \int_0^{2\pi} \int_0^\infty G(r, \theta) F(r) dr d\theta &= \int_0^{2\pi} d\theta \int_0^\infty -F''(r) dr \\ &= 2\pi \lim_{R \rightarrow \infty} \{-F'(R) + F'(0)\} = 0. \end{aligned}$$

2. PROOF OF THEOREM

2.1. (see [1, 2, 6]). Let M be a complete noncompact Riemannian manifold and let L be a ray in M , i.e., the image of an isometric map $\gamma: [0, \infty) \rightarrow M$.

Define a function $f_L: M \rightarrow \mathbf{R}$ by $f_L(\cdot) = \lim_{t \rightarrow \infty} \{d(\cdot, \gamma(t)) - t\}$. We call the function f_L the *Busemann function of L* . The Busemann functions are continuous on M . Suppose a sequence $\{p_n\} \subset M$ converges to a point $p \in M$ and a sequence $\{T_n\}$ of minimizing geodesics connecting p_n to $\gamma(t_n)$ converges to a ray L_1 . We call such an L_1 a *co-ray from p to L* . We say that a co-ray L_1 to L is *maximal* if any extension of L_1 as a geodesic is not a co-ray to L . We say that L_1 is an *asymptote to L* if the maximal co-ray L_1 to L is a straight line, i.e., the image of an isometric map $\alpha: (-\infty, \infty) \rightarrow M$. Let $C(L)$ be the set of all origins of maximal co-rays to L . It is known that for any $p \notin C(L)$ there exists the unique co-ray L_1 from p to L and the Busemann function f_L is differentiable at p with gradient vector $-\dot{\gamma}_1(0)$, where $\gamma_1: [0, \infty) \rightarrow M$ is a representation of L_1 . Suppose the co-ray relation is an equivalence relation. Then, $f_L - f_{L_1}$ is constant on M for any L and L_1 which are co-rays to each other if f_L and f_{L_1} are differentiable on $f_L^{-1}(-\infty, a]$ for some $a \in \mathbf{R}$. It should be noted that $C(L) = \phi$ is not assumed.

2.2.. (see [1, 3]). Let $M = (\mathbf{R}^2, g)$ be a complete Riemannian manifold and let L be a ray with representation $\gamma: [0, \infty) \rightarrow M$. Let $\{p_n\}$ be a sequence converging to a point $p \in M$. We denote by $A(q, t)$ the greatest lower bound of all angles of L and the minimizing geodesics from $q \in M$ to $\gamma(t)$ at $\gamma(t)$. Then, there is a sequence $\{t_n\}$ such that $t_n \rightarrow \infty$ and $A(p_n, t_n) \rightarrow 0$ as $n \rightarrow \infty$. If M has the total curvature and a straight line L in M , then the total curvature of each component of $M - L$ is nonpositive.

Let L_1 be a co-ray from p to a ray L and let A be a minimizing geodesic connecting p and the origin q of L . Let D be the domain bounded by A , L and L_1 with convex angles α_1 and α_2 at p and q , respectively. If the total curvature $\varepsilon(D)$ exists, then

$$\varepsilon(D) = \alpha_1 + \alpha_2 - \pi$$

because of the Gauss-Bonnet formula. When all vertices are in infinity, we get the following. Let L_1 be a straight line containing a co-ray to a straight line L and let L_2 be a straight line such that it contains a co-ray to L_- and L_{2-} contains a co-ray to L_{1-} , where L_- is the reversed geodesic of L . If D is the domain bounded by L , L_1 and L_2 and the total curvature $\varepsilon(D)$ of D exists, then $\varepsilon(D) \leq -\pi$.

2.3. *Proof of theorem* We assume that M is a manifold as in the Theorem. Let L be a ray in $M - K$ and f_L the Busemann function of L . Assume that $m_1 < f_L(q) < m_2$ for any $q \in K$. Then, all levels $f_L^{-1}(a)$, $a < m_1$, of f_L are geodesics because $f_L^{-1}(-\infty, m_1]$ is flat and all geodesics in $M - K$ are minimizing, i.e., all metric balls are isometric to ones in the Euclidean plane. The geodesic is not a closed curve, and, in particular, $f_L^{-1}(-\infty, m_1]$ is isometric to the closed Euclidean half-plane with boundary L_1 . By the same argument, we know that $f_L^{-1}(-\infty, m_1] \cup f_{L_1}^{-1}(-\infty, m_3] \cup f_{L_{1-}}^{-1}(-\infty, m_4]$ are flat

for some m_3 and m_4 . Let $L_2 = f_{L_{1+}}^{-1}(m_3)$ and $L_3 = f_{L_{1-}}^{-1}(m_4)$. Then, since $f_L^{-1}(-\infty, m_1]$ is the Euclidean half-plane, L_2 and L_3 are both asymptotes to L with suitable orientation, say L_{2+} and L_{3+} .

We now want to prove that L_{2-} is a co-ray to L_{3-} . If this is not true, we have two possibilities; (1) all co-rays to L_{3-} from all points of L_{2-} going to infinity intersect K , and (2) there is a co-ray L_0 to L_{3-} from some point p_0 of L_{2-} such that it does not intersect K and is not a subray of L_{2-} . Suppose (1) occurs. Let $\{p_n\} \subset L_{2-}$ be a sequence going to infinity and let B_n be a co-ray from p_n to L_{3-} for each n . Since all B_n intersect K , there is a subsequence of $\{B_n\}$ converging to a straight line A . Then, A is such that A_+ contains a co-ray to L_{3-} and A_- contains a co-ray to L_{2-} . Hence, the total curvature $\varepsilon(D)$ of the domain D bounded by L_2 , L_3 and A is less than or equal to $-\pi$ because of 2.2. Since the total curvature of the union $M - D$ of three half-plane is nonpositive, we have that $\varepsilon(M) \leq -\pi$, a contradiction. Suppose (2) occurs. Let D be the domain bounded by L_3 , L_0 and the subray of L_{2+} from p_0 . The total curvature $\varepsilon(D)$ of D is $\alpha - \pi < 0$, where α is the angle of D at p_0 , because of the Gauss-Bonnet formula. Since $K \subset D$, the total curvature of $M - D$ is zero. Hence, $\varepsilon(M) < 0$, a contradiction. Thus, we proved that L_{2-} is a co-ray to L_{3-} . In particular, if $m_5 < f_{L_{2-}}(q)$ for $q \in K$, then L_{2-} and L_{3-} are parallel in the Euclidean half-plane $f_{L_{2-}}^{-1}(-\infty, m_5]$. We assume that the distance between L_{2-} and L_{3-} is l_2 .

In order to prove that the support of $g - g_0$ is compact, it suffices to prove that $l_2 = l_1$, where l_1 is the distance between L_{2+} and L_{3+} in the Euclidean half-plane $f_L^{-1}(-\infty, m_1]$. Let L_4 be a straight line in $f_{L_{2-}}^{-1}(-\infty, m_5]$ which contains a minimal geodesic from L_{3-} to L_{2-} . Then, by Euclidean geometry in $f_{L_{1+}}^{-1}(-\infty, m_3]$, L_{4+} is an asymptote to L_{1+} , and, moreover, $f_{L_{1+}} - f_{L_{4+}} = \text{const.}$ on M , because $f_{L_{i+}}$, $i = 1, 4$ are differentiable in $f_{L_{1+}}^{-1}(-\infty, m_3]$ and the co-ray relation for L_{1+} is transitive. The same is true for L_{1-} and L_{4-} . Suppose for the indirect proof that $l_2 \neq l_1$, say $l_2 > l_1$. Let $p = L_3 \cap L_4$, $f_{L_{4+}}(p) = a$ and $f_{L_{4-}}(p) = b$. L_3 is a level of $f_{L_{4-}}$. Since L_4 is a straight line, $f_{L_{4+}}^{-1}(a)$ cannot have any interior point of $f_{L_{4-}}^{-1}(-\infty, b]$. On the other hand, the point of L_1 backward on L_{1+} with distance l_2 from $L_2 \cap L_1$ is contained in both $f_{L_{4+}}^{-1}(a)$ and is an interior point $f_{L_{4-}}^{-1}(-\infty, b]$, a contradiction. The Theorem is proved.

REFERENCES

1. H. Busemann, *The Geometry of geodesics*, Academic press, New York, 1955.
2. —, *Recent synthetic differential geometry*, Springer, Berlin-Heidelberg-New York, 1970.
3. S. Cohn-Vossen, *Totalkrümmung und geodätische Linien auf einfach zusammenhängenden, offenen, vollständigen Flächenstücken*, Mat. Sb. (N.S.) **1** (1936), 139–164.
4. L. W. Green and R. Gulliver, *Planes without conjugate points*, J. Differential Geom. **22** (1985), 43–47.

5. R. E. Greene and H. Wu, *Gap theorems for noncompact Riemannian manifolds*, Duke Math. J. **49** (1982), 731–756.
6. N. Innami, *Differentiability of Busemann functions and total excess*, Math. Z. **180** (1982), 235–247.
7. —, *The n -plane with integral curvature zero and without conjugate points*, Proc. Japan Acad. Ser. A **62** (1986), 282–284.
8. Y. Nasu, *On asymptotes in a metric space with non-positive curvature*, Tôhoku Math. J. **9** (1957), 68–95.

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