THE SIMPLE-ZERO THEOREM FOR SUPPORT POINTS IN Σ

Y. J. LEUNG AND G. SCHOBER

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ABSTRACT. The purpose of this article is to confirm a simple-zero conjecture raised by the second author for support points in Σ . The conjecture implies that the omitted arcs of an extremal function in Σ for a linear problem can branch out in, at most, three equiangular directions at a time.

I. INTRODUCTION

Let Σ be the class of functions $f(z) = z + b_0 + b_1/z + \cdots$ analytic and univalent in $\Delta = \{z: |z| > 1\}$. The Koebe function $K(z) = z + 2 + 1/z$, which omits the line segment $[0, 4]$ from its range, is a prominent member of Σ . Let L be a continuous linear functional defined on the space of functions analytic in Δ , with the topology of locally uniform convergence. In addition, assume that L is nonconstant on Σ and that $L(1) = 0$. A function f in Σ is called a support point of Σ for L if $\operatorname{Re} L(f) \geq \operatorname{Re} L(g)$ for all g in Σ . It follows from Schiffer's boundary variation (cf. [7]) that the continuum Γ omitted by a support point f consists of analytic arcs lying on trajectories of the quadratic differential $L(1/(f - w))dw^2$.

As a function of w , $L(1/(f - w))$ has an analytic extension to some open region containing the omitted set Γ . Thus the structure theory of quadratic differentials [9] shows that the analytic arcs of Γ are joined to one another only at the zeros of $L(1/(f - w))$. At a zero of order $k \geq 1$, at most $k + 2$ arcs may join at a time in a subset of $k + 2$ equiangular directions. For example, the coefficient functional $L(f) = b_3$ leads to a quadratic differential of the form $(w^2 - b_1)dw^2$. The omitted set of a support point for this functional forks at each of the two simple zeros at $\pm\sqrt{b_1}$ in three equiangular directions. Further examples of support points can be found in [3, 4].

Based on various known examples, the second author of this article conjectured [4, 8] that the quadratic differential $L(1/(f - w))dw^2$ associated with a support point f in Σ can have only simple zeros on the set Γ omitted by f .

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It is the purpose of this article to verify this simple-zero conjecture. Our main result is:

Theorem 1. *If f is a support point in Σ for some linear functional L , then its associated quadratic differential $L(1/(f-w))dw^2$ can have only simple zeros on the omitted set of f .*

Geometrically, this theorem implies that the omitted arcs can fork at the simple zeros in, at most, three equiangular directions. Some supporting evidence had been presented recently in [4, 5]. For example, it is known that if Γ contains a zero of order $k \geq 2$ at w_0 , then it cannot contain a subset of arcs branching out symmetrically at w_0 . On the other hand, it should be pointed out that $L(1/(f-w))$ may have zeros of any order off of Γ [5, Theorem 5.1]. In addition, a result of Abu-Muhanna [1] shows that *any* single analytic slit of mapping radius one is a support point for some linear functional L .

The proof of Theorem 1 consists of two parts. First, we use a technique of Schiffer's [2, pp. 216–218] to convert the conjecture into a coefficient problem. Next, we present a form of the second variation of the Koebe function to solve this coefficient problem.

II. SCHIFFER'S TRUNCATION VARIATION

For completeness, we shall present Schiffer's truncation variation (cf. also [4, pp. 57–58]) in order to convert the simple-zero conjecture into a coefficient problem for functions analytic and univalent in the complement of the line segment $[0, 4]$.

After a translation, we may assume that $w = 0$ is omitted by a support point f for a functional L and is a zero of order k for the function $L(1/(f-w))$. Expanding $L(1/(f-w))$ about $w = 0$, we see that $L(f^{-j}) = 0$ for $j = 0, 1, \dots, k$ and $L(f^{-k-1}) \neq 0$. Furthermore, after rotating the functional and the function, we may assume also that $L(f^{-k-1}) > 0$ and that one of the omitted arcs emanates from the origin in the positive horizontal direction.

To apply Schiffer's truncation, we consider a disk $|w| \leq \delta$ for a sufficiently small positive δ . We delete all the arcs of Γ except for that connected part of the arc emanating from the origin in the positive horizontal direction that lies inside the disk. Designate this subarc by Γ_δ , and let F_δ be the conformal map from Δ onto the complement of Γ_δ with an expansion of the form $F_\delta(z) = \rho z + O(1)$ around infinity and $\rho > 0$. Then F_δ and the original f are related by a Schwarz function $\omega: \Delta \rightarrow \Delta$ satisfying $F_\delta \circ \omega = f$ and $\omega'(\infty) = 1/\rho$. As $\delta \rightarrow 0$, we have $\rho \rightarrow 0$. The function $f_\delta = (1/\rho)F_\delta$ is a support point in Σ for the functional L_δ defined by $L_\delta(g) = L(\rho g \circ \omega)$. Let us designate the set omitted by f_δ to be γ_δ . It is obtained from dilating Γ_δ by the factor $1/\rho$. Since γ_δ is an analytic arc emanating from $w = 0$ in the positive horizontal direction, it approaches the line segment $[0, 4]$ as $\delta \rightarrow 0$. By the Carathéodory convergence theorem, f_δ approaches the Koebe function $K(z) = z + 2 + 1/z$ as $\delta \rightarrow 0$.

Let F be analytic and univalent in the complement of γ_δ and have an expansion of the form $F(w) = w + \sum_{j=0}^{\infty} C_j(\delta)w^{-j}$ around infinity. Then $f_\delta^* = F \circ f_\delta$ serves as a variation of f_δ , and so $\operatorname{Re} L_\delta(f_\delta^*) \leq \operatorname{Re} L_\delta(f_\delta)$. Using the definition of L_δ , we rewrite this inequality as $\operatorname{Re} L(f_\delta^* \circ \omega) \leq \operatorname{Re} L(f_\delta \circ \omega)$. Since $f_\delta^* \circ \omega = f_\delta \circ \omega + \sum_{j=0}^{\infty} C_j(\delta)\rho^j f^{-j}$ for $|f| > \delta$, we conclude that $\operatorname{Re}\{\sum_{j=0}^{\infty} C_j(\delta)\rho^j L(f^{-j})\} \leq 0$ when δ is sufficiently small. From the assumption on the order of the zero at $w = 0$, the terms of this series with index $j \leq k$ are zero. Therefore, we have

$$\operatorname{Re}\{C_{k+1}(\delta)\}\rho^{k+1}L(f^{-k-1}) + O(\rho^{k+2}) \leq 0,$$

where $L(f^{-k-1})$ is positive. Dividing by ρ^{k+1} and letting $\delta \rightarrow 0$ leads us to the conclusion that $\operatorname{Re} C_{k+1} \leq 0$ for any univalent function $F(w) = w + \sum_{j=0}^{\infty} C_j w^{-j}$ in the complement of $\gamma_0 = [0, 4]$. Hence, the presence of a k th order zero implies the following:

If the omitted set Γ of a support point for some functional contains a k th order zero of its associated quadratic differential $L(1/(f-w))dw^2$, then the inequality

$$(1) \quad \operatorname{Re} C_{k+1} \leq 0$$

is valid for every function F analytic and univalent in the complement of the line segment $[0, 4]$ with expansion $F(w) = w + \sum_{j=0}^{\infty} C_j w^{-j}$ in a neighborhood of infinity.

In the next section, for each $k \geq 2$ we shall use a second-order variation of the identity to show that there are functions F with $k+1$ st coefficient C_{k+1} satisfying $\operatorname{Re} C_{k+1} > 0$. Combined with the statement above, this eliminates the presence of all k th order zeros of $L(1/(f-w))dw^2$ in the omitted set for $k \geq 2$.

III. SECOND-ORDER VARIATION

To derive variational formulas for mappings defined in the complement of the segment $[0, 4]$, we shall use the Loewner equation for the familiar class S of univalent analytic functions f defined in the unit disk $\mathbb{D} = \{z: |z| < 1\}$, with $f(0) = 0$ and $f'(0) = 1$. For simplicity, let $\kappa = \kappa(t)$ be a continuous complex-valued function on $[0, \infty)$ with $|\kappa(t)| = 1$. Then the differential equation

$$(2) \quad \frac{\partial f}{\partial t} = z \left(\frac{1 + \kappa z}{1 - \kappa z} \right) \frac{\partial f}{\partial z}, \quad t \geq 0, \quad z \in \mathbb{D},$$

has a solution $f(z, t)$ such that the functions $e^{-t}f(\cdot, t)$ belongs to S (cf. [6, §6.1]). In particular, $f(\cdot, 0)$ belongs to S .

In order to refer the Loewner chain $f(\cdot, t)$ to the complement of the segment $[0, 4]$, we shall use the function $K(z) = z + 2 + 1/z$, which also maps $\mathbb{D} \setminus \{0\}$ onto the complement of $[0, 4]$. Denote the inverse of this mapping by S .

Then we have $S(w) = \frac{1}{2}(w - 2 - w\sqrt{1 - 4/w})$, where the branch of the square root is chosen so that $\sqrt{1} = 1$. The corresponding Loewner chain of mappings $F(\cdot, t)$ in the complement of $[0, 4]$ are defined by $F(w, t) = 1/f(S(w), t)$. From (2) and the chain rule one finds that they satisfy the differential equation

$$(3) \quad \frac{\partial F}{\partial t} = \left(S(w) - \frac{1}{S(w)} \right) \left(\frac{1 + \kappa S(w)}{1 - \kappa S(w)} \right) \frac{\partial F}{\partial w}, \quad t \geq 0,$$

and $e^t F(w, t) = w + O(1)$ as $w \rightarrow \infty$. In particular, $F(\cdot, 0)$ is a competitor for the inequality (1).

We choose $\kappa(t) = e^{i\varepsilon\theta(t)}$, with ε being a small real parameter and θ a bounded continuous function of t . Then we have the expansions $\kappa = 1 + i\varepsilon\theta - \frac{1}{2}\varepsilon^2\theta^2 + O(\varepsilon^3)$ and

$$\frac{1 + \kappa S}{1 - \kappa S} = \frac{1 + S}{1 - S} + i\varepsilon\theta \frac{2S}{(1 - S)^2} - \varepsilon^2\theta^2 \frac{S(1 + S)}{(1 - S)^3} + O(\varepsilon^3)$$

as $\varepsilon \rightarrow 0$. In this case, the differential equation (3) is of the form

$$(4) \quad \frac{\partial F}{\partial t} = \left[-\frac{(1 + S)^2}{S} - 2i\varepsilon\theta \frac{1 + S}{1 - S} + \varepsilon^2\theta^2 \left(\frac{1 + S}{1 - S} \right)^2 + O(\varepsilon^3) \right] \frac{\partial F}{\partial w}$$

for the function

$$F(w, t, \varepsilon) = \phi(w, t) + \varepsilon V(w, t) + \varepsilon^2 Q(w, t) + O(\varepsilon^3).$$

Since $(1 + S)^2/S = w$, the differential equation (4) for $\varepsilon = 0$ becomes $\partial\phi/\partial t = -w\partial\phi/\partial w$. It follows that $e^t\phi(w, t) = w$, and so the functions $e^t F(\cdot, t, \varepsilon)$ are normalized variations of the identity mapping.

A comparison of the coefficients of ε and ε^2 in (4) leads to the differential equations

$$\begin{aligned} \frac{\partial V}{\partial t} &= -w \frac{\partial V}{\partial w} - 2ie^{-t}\theta \frac{1 + S}{1 - S}, \\ \frac{\partial Q}{\partial t} &= -w \frac{\partial Q}{\partial w} - 2i\theta \frac{1 + S}{1 - S} \frac{\partial V}{\partial w} + e^{-t}\theta^2 \left(\frac{1 + S}{1 - S} \right)^2, \end{aligned}$$

for the first and second variations. By expressing S explicitly in terms of w , these equations can be rewritten as

$$(5) \quad \begin{aligned} \frac{\partial V}{\partial t} &= -w \frac{\partial V}{\partial w} - 2ie^{-t}\theta \frac{1}{\sqrt{1 - 4/w}}, \\ \frac{\partial Q}{\partial t} &= -w \frac{\partial Q}{\partial w} - 2i\theta \frac{1}{\sqrt{1 - 4/w}} \frac{\partial V}{\partial w} + e^{-t}\theta^2 \frac{1}{1 - 4/w}. \end{aligned}$$

Since $F(w, t, \varepsilon) - \phi(w, t) = O(1)$ as $w \rightarrow \infty$, the expansions of V and Q around infinity are of the form $V(w, t) = \sum_{n=0}^{\infty} V_n(t)w^{-n}$ and $Q(w, t) = \sum_{n=0}^{\infty} Q_n(t)w^{-n}$. It follows from the system of differential equations (5) that

$$V'_n = nV_n - 2ie^{-t}\theta d_n$$

and

$$Q'_n = nQ_n + 2i\theta \sum_{j=1}^{n-1} j d_{n-1-j} V_j + 4^n e^{-t} \theta^2,$$

where

$$\frac{1}{\sqrt{1-4/w}} = \sum_{n=0}^{\infty} d_n w^{-n} \quad \text{with} \quad d_n = \frac{(2n)!}{(n!)^2}.$$

If w is restricted, say, to a large circle, then $e^t F(w, t, \varepsilon)$ is uniformly bounded. Therefore the coefficients $V_n(t)$ and $Q_n(t)$ vanish at $t = \infty$. Thus this system can be integrated to yield

$$(6) \quad V_n(0) = 2i d_n \int_0^{\infty} e^{-(n+1)t} \theta(t) dt$$

and

$$\begin{aligned} Q_n(0) &= -4^n \int_0^{\infty} e^{-(n+1)t} \theta^2(t) dt \\ &+ 4 \sum_{j=1}^{n-1} j d_j d_{n-1-j} \int_0^{\infty} \int_t^{\infty} e^{-(n-j)t-(j+1)s} \theta(t) \theta(s) ds dt. \end{aligned}$$

In the expansion of $F(w, 0, \varepsilon) = w + \sum_{n=0}^{\infty} C_n(0, \varepsilon) w^{-n}$, the coefficients are of the form $C_n(0, \varepsilon) = \varepsilon V_n(0) + \varepsilon^2 Q_n(0) + O(\varepsilon^3)$. Evidently, $V_n(0)$ is purely imaginary and $Q_n(0)$ is real, so $\operatorname{Re} C_n(0, \varepsilon) = \varepsilon^2 Q_n(0) + O(\varepsilon^3)$. For each $n \geq 3$, we shall find a function θ such that $Q_n(0) > 0$. This will show the existence of a function $F(\cdot, 0, \varepsilon)$, nearby the identity function, with $\operatorname{Re} C_n(0, \varepsilon)$ positive. As pointed out at the end of §II, this will complete the proof of Theorem 1.

Substitute $\theta(t) = e^{(2-\delta)t}$ into (6), and discard from the sum all terms except the one for which $j = 1$. Then we have

$$\begin{aligned} Q_n(0) &\geq -4^n \int_0^{\infty} e^{-(n-3+2\delta)t} dt + 4d_1 d_{n-2} \int_0^{\infty} \int_t^{\infty} e^{-(n-3+\delta)t-\delta s} ds dt \\ &= \frac{1}{n-3+2\delta} \left[-4^n + \frac{4d_1 d_{n-2}}{\delta} \right] \end{aligned}$$

for $n > 3 - 2\delta$, where d_1 and d_{n-2} are positive. When $n \geq 3$ and δ is positive and sufficiently small, it follows that $Q_n(0) > 0$. Although this is the desired conclusion, the function θ that was chosen is not bounded. However, the truncation $\theta_N(t) = \min\{N, \theta(t)\}$ is bounded, and an application of the dominated convergence theorem leads to the same conclusion that $Q_n(0)$ is positive for some θ_N .

Thus for each $n \geq 3$, we have shown the existence of a function F , analytic and univalent in the complement of $[0, 4]$, such that the coefficient C_n in the expansion $F(w) = w + \sum_{j=0}^{\infty} C_j w^{-j}$ around $w = \infty$ satisfies $\operatorname{Re} C_n > 0$. Therefore we have eliminated the presence of $n-1$ st order zeros for each $n \geq 3$. This completes the proof of Theorem 1.

Remark. The search for a positive $Q_n(0)$ can be rephrased as an eigenvalue problem for an integral equation. After a change of variables, one can ask for the maximum of

$$\frac{-4^n \int_0^1 y^n \vartheta^2(y) dy + 4 \sum_{j=1}^{n-1} j d_j d_{n-1-j} \int_0^1 \int_0^y x^j y^{n-1-j} \vartheta(x) \vartheta(y) dx dy}{\int_0^1 \vartheta^2(y) dy}$$

for functions ϑ , say, in $\mathcal{L}^2[0, 1]$. If λ denotes this maximum, then the Euler equation for this problem is the integral equation

$$-4^n y^n \vartheta(y) + 2 \sum_{j=1}^{n-1} j d_j d_{n-1-j} \left\{ y^{n-1-j} \int_0^y x^j \vartheta(x) dx + y^j \int_y^1 x^{n-1-j} \vartheta(x) dx \right\} = \lambda \vartheta(y).$$

The range of the function $-4^n y^n$ and, hence, the continuous spectrum of the integral equation is the interval $[-4^n, 0]$. Implicit in our work is the existence of a positive eigenvalue.

REFERENCES

1. Y. Abu-Muhanna and Y. J. Leung, *On analytic slit mappings in the class Σ* , Proc. Amer. Math. Soc. **99** (1987), 44–48.
2. A. Chang, M. M. Schiffer and G. Schober, *On the second variation for univalent functions*, J. Analyse Math. **40** (1981), 203–238.
3. W. E. Kirwan and G. Schober, *New inequalities from old ones*, Math. Z. **180** (1982), 19–40.
4. Y. J. Leung and G. Schober, *Low order coefficient estimates in the class Σ* , Ann. Acad. Sci. Fenn. Ser. AI Math. **11** (1986), 36–61.
5. —, *On the structure of support points in the class Σ* , J. Analyse Math. **46** (1986), 176–193.
6. Chr. Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.
7. G. Schober, *Univalent functions—Selected topics*, Lecture notes in math. #478, Springer-Verlag, Berlin, 1975.
8. —, *Some conjectures for the class Σ* , in Topics in complex analysis, Contemporary Math. **38** (1985), 13–21.
9. K. Strebel, *Quadratic differentials*, Springer-Verlag, Berlin, 1984.

UNIVERSITY OF DELAWARE, NEWARK, DELAWARE 19716

INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47405