REDUCTION OF LOCAL LIE ALGEBRA STRUCTURES

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ABSTRACT. We deal with the reduction theory of local Lie algebra structures in the function space through coadjoint equivariant momentum mappings.

1

The Marsden-Weinstein symplectic reduction theory through momentum mappings [4] seems to be an epoch-making and remarkable theory in symplectic geometry. There are too many works which have applied that theory to refer to them all here. Marsden and Ratiu [3] gave a generalization of Marsden-Weinstein reduction theory in the Poisson manifold case. Also, Albert [1] has generalized Marsden-Weinstein reduction theory in the contact manifold case.

A *local Lie algebra structure* [2] in a function space is a more general idea than that of symplectic, Poisson or contact structure. It is also called a *Jacobi structure* by Lichnerowicz and coworkers. Since a Poisson manifold is a union of symplectic manifolds which fit together in a smooth way [6], it can be expected in some cases that a Poisson manifold divided by some relation or symmetry is a symplectic manifold. In the local Lie algebra structure case, it is natural to ask if a local Lie algebra structure divided by some relation is again a local Lie algebra, Poisson, or symplectic structure. In this note, we shall study reduction theory through momentum mappings of local Lie algebra structures.

2

Let P be a C^∞ -manifold. A local Lie algebra structure in $C^\infty(P)$ is a bracket operation $\{ , \}$ in $C^\infty(P)$ satisfying:

- (1) $\{ , \}$ defines a Lie algebra structure in $C^\infty(P)$ over $\mathbf{R}$.
- (2) $\{f, h\}$ is continuous both in f and h with respect to the C^∞ -topology, and

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(3) $\text{supp}\{f, h\} \subset \text{supp } f \cap \text{supp } h$ for each f and h , where $\text{supp } f$ = the support of a function f (cf. [2]).

It is shown in [2] that each local Lie algebra structure in $C^\infty(P)$ corresponds to a unique pair (ξ, Ξ) consisting of a vector field ξ and a 2-vector field Ξ on P as follows:

$$\begin{aligned}\{f, h\} &= f(\xi] dh) - h(\xi] df) + \langle \Xi | df \wedge dh \rangle \\ &= f[\xi, h] - h[\xi, f] - [[\Xi, f], h],\end{aligned}$$

where $[,]$ denotes the Schouten bracket (cf. [5]). ξ and Ξ must satisfy

$$[\xi, \Xi](= \mathcal{L}_\xi \Xi) = 0 \quad \text{and} \quad [\Xi, \Xi] + 2\xi \wedge \Xi = 0.$$

Remarks (cf. [2]). (1) If $\xi = 0$, the local Lie algebra structure is a Poisson structure.

(2) If $\xi = 0$ and $\text{rank } \Xi = \dim P$, the local Lie algebra structure is the Poisson structure induced from the symplectic structure Ξ^{-1} .

(3) If ξ and $\{[f, \Xi] \mid f \in C^\infty(P)\}$ span the tangent space of an odd-dimensional manifold P at each point, then $\{ , \}$ is the Lagrange bracket of a contact structure.

3

As an analogy of the Poisson reduction theorem [3], we would like to consider the reduction problem for local Lie algebra structures. Let $i: M \subset P$ be a submanifold of P . Let $\mathcal{E}$ be a subbundle of $TP|_M$. Assume that

- (1) $\mathcal{E} \cap TM$ is an integrable subbundle of $TP|_M$,
- (2) the foliation Φ corresponding to $\mathcal{E} \cap TM$ is regular, that is, M/Φ is a C^∞ -manifold and the canonical projection $\pi: M \rightarrow M/\Phi$ is a C^∞ -submersion, and
- (3) $[\xi, \mathcal{E}^0] \subset \mathcal{E}^0$ and $[[\Xi, \mathcal{E}^0], \mathcal{E}^0] \subset \mathcal{E}^0$, where $\mathcal{E}^0 := \{F \in C^\infty(P) \mid v] dF = 0 \text{ for each } v \in \mathcal{E}\}$.

Borrowing from Marsden-Ratiu's work, we say that $(P, M, \mathcal{E})$ is reducible if M/Φ has a local Lie algebra structure $\{ , \}_\Phi$ such that for any C^∞ -functions f and h on M/Φ and smooth extensions F and H of $f \circ \pi$ and $h \circ \pi$ with $F, H \in \mathcal{E}^0$, we have $\{F, H\} \circ i = \{f, h\}_\Phi \circ \pi$.

Proposition. $(P, M, \mathcal{E})$ is reducible if and only if $[\xi, (\mathcal{E} + TM)^0] = 0$ and $[[\Xi, \mathcal{E}^0], (\mathcal{E} + TM)^0] = 0$ on M .

Proof. Take $F \in (\mathcal{E} + TM)^0$. Then there is a constant function, say c , on M/Φ such that $c \circ \pi = F \circ i$. We get

$$\begin{aligned}0 &= \{1, c\}_\Phi \circ \pi = \{1, F\} \circ i \\ &= ([\xi, F] - [[\Xi, 1], F]) \circ i = [\xi, F] \circ i.\end{aligned}$$

Take $H \in \mathcal{E}^0$. Then there is a function h such that $H \circ i = h \circ \pi$. Let F_0 be $F - c$. Then $F_0 \in (\mathcal{E} + TM)^0$ and $F_0 \circ i = 0$. Now, we have that

$$\begin{aligned}\{F, H\} \circ i &= \{F_0 + c, H\} \circ i = \{F_0, H\} \circ i + \{c, H\} \circ i \\ &= \{F_0, H\} \circ i + \{c, h\}_{\Phi} \circ \pi.\end{aligned}$$

Since $\{F, H\} \circ i = \{c, h\}_{\Phi} \circ \pi$, we get that $\{F_0, H\} \circ i = 0$. So, it holds that

$$\begin{aligned}0 &= \{F_0, H\} \circ i \\ &= (F_0[\xi, H] - H[\xi, F_0] - [[\Xi, F_0], H]) \circ i \\ &= (F_0 \circ i)[\xi, H] \circ i - (H \circ i)[\xi, F_0] \circ i \\ &\quad - ([[\Xi, F_0], H]) \circ i \\ &= - ([[\Xi, F_0], H]) \circ i.\end{aligned}$$

Conversely, for $f, h \in C^\infty(M/\Phi)$, take $F, H \in \mathcal{E}^0$ such that $F \circ i = f \circ \pi$ and $H \circ i = h \circ \pi$. We have that

$$\begin{aligned}[\mathcal{E}, \{F, H\}] &= [\mathcal{E}, F][\xi, H] + F[\mathcal{E}, [\xi, H]] \\ &\quad - [\mathcal{E}, H][\xi, F] - H[\mathcal{E}, [\xi, F]] \\ &\quad - [\mathcal{E}, [[\Xi, F], H]] \\ &= 0\end{aligned}$$

from assumption (3). This means that $\{F, H\}$ is constant on leaves of Φ . So, define $\{f, h\}_{\Phi}$ by $\{f, h\}_{\Phi} \circ \pi := \{F, H\} \circ i$. To check that $\{, \}_{\Phi}$ is well defined, take $H' \in \mathcal{E}^0$ such that $H' \circ i = h \circ \pi$. Then $H - H' \in (\mathcal{E} + TM)^0$ and $(H - H') \circ i = 0$. Therefore, we have

$$\begin{aligned}\{F, H - H'\} \circ i &= F \circ i[\xi, H - H'] \circ i \\ &\quad - (H - H') \circ i[\xi, F] \circ i \\ &\quad - [[\Xi, F], H - H'] \circ i \\ &= 0.\end{aligned}$$

It is easy to check skew-symmetry, Jacobi's identity and locality of $\{, \}_{\Phi}$. $\square$

4

By analogy to Hamiltonian vector fields on symplectic manifolds or infinitesimal contact transformations on contact manifolds, we consider a linear map $\beta: C^\infty(P) \rightarrow \mathcal{X}(P)$ defined by

$$\beta(f) := f\xi - [f, \Xi].$$

β is a Lie algebra homomorphism of $C^\infty(P)$ into $\mathcal{X}(P)$. So, we can now consider momentum mappings for a local Lie algebra structure.

Definition [5]. Let a Lie group G act on P and preserve the local Lie algebra structure which is defined by (ξ, Ξ) . A map $J: P \rightarrow \mathfrak{g}^*$ is a momentum mapping of the action of G if $(\beta \circ \hat{J})(\zeta) = -\zeta_p$ for each $\zeta \in \mathfrak{g}$, where ζ_p is the fundamental vector field on P defined by the action of G and $\hat{J}: \mathfrak{g} \rightarrow C^\infty(P)$ is defined by $\hat{J}(\zeta)(x) = \langle \zeta, J(x) \rangle$ and is called the dual of J or the co-momentum mapping of J .

The importance of momentum mappings in symplectic or Poisson geometry seems to stand on Noether's theorem. In our local Lie algebra structure case, we have

Noether's theorem [5]. Let f be a G -invariant function on P . Then $\hat{J}(\zeta)$ is a first integral of $\beta(f)$ for each $\zeta \in \mathfrak{g}$ if f is invariant by ξ .

Now we shall study local Lie algebra structure reduction through an equivariant momentum mapping. Concerning the equivariance of momentum mappings of local Lie algebra structures, we refer to [5]. Let $J: P \rightarrow \mathfrak{g}^*$ be a coadjoint equivariant momentum mapping, and let $\mu \in \mathfrak{g}^*$ be a clean value of J , i.e., μ is a regular value and $T_x J^{-1}(\mu) = \text{Ker } T_x J$ for each $x \in J^{-1}(\mu)$. Let $\mathcal{E}$ be the collection of tangent vectors to the G -orbits and let M be $J^{-1}(\mu)$. Assume that

(1) $\mathcal{E} \cap TM = \{\eta_p(x) \mid x \in M, \eta \in \mathfrak{g}_\mu\}$ is a subbundle of $TP|_M$.

(2) The foliation Φ corresponding to $\mathcal{E} \cap TM$ is regular.

It is true that $\mathcal{E} \cap TM$ is integrable, $[\xi, \mathcal{E}^0] \subset \mathcal{E}^0$, and $[[\Xi, \mathcal{E}^0], \mathcal{E}^0] \subset \mathcal{E}^0$. Then we have the following theorem.

Theorem. Under the assumptions above, $(P, J^{-1}(0), \mathcal{E})$ is reducible.

Lemma. (1) ξ is tangent to $M = J^{-1}(\mu)$, and

(2) $[\xi, (\mathcal{E} + TM)^0] = 0$ on M .

Proof. To show $TJ(\xi) = 0$, we deal with $\langle TJ(\xi), \zeta \rangle = \langle \xi, \hat{J}(\zeta) \rangle$ for $\zeta \in \mathfrak{g}$. Since $-\zeta_p = \beta \hat{J}(\zeta) = \hat{J}(\zeta)\xi - [\hat{J}(\zeta), \Xi]$ preserves the local Lie algebra structure, we have

$$0 = -[\zeta_p, \xi] = [\hat{J}(\zeta), \xi]\xi - [[\hat{J}(\zeta), \xi], \Xi]$$

and

$$0 = -[\zeta_p, \Xi] = [\hat{J}(\zeta), \xi]\Xi.$$

Fix an arbitrary point $x \in P$. If $\Xi_x \neq 0$, the second equation implies $[\hat{J}(\zeta), \xi]_x = 0$. If $\Xi_x = 0$, then the first equation implies that $[\hat{J}(\zeta), \xi]_x \xi_x = 0$, and we have $[\hat{J}(\zeta), \xi]_x = 0$. Therefore, we have $[\hat{J}(\zeta), \xi] = 0$ for each $\zeta \in \mathfrak{g}$. $\square$

Proof of theorem. Take $F \in (\mathcal{E} + TM)^0$ and $H \in \mathcal{E}^0$. We have

$$\begin{aligned} \langle TJ[\Xi, H], \zeta \rangle &= [[\Xi, H], \hat{J}(\zeta)] = -[[\Xi, \hat{J}(\zeta)], H] \\ &= [\beta \hat{J}(\zeta) - \hat{J}(\zeta)\xi, H] = [-\zeta_p, H] - \hat{J}(\zeta)[\xi, H] = -\hat{J}(\zeta)[\xi, H] \end{aligned}$$

for each $\zeta \in \mathfrak{g}$. If $\mu = 0 \in \mathfrak{g}^*$, then $\hat{J}(\zeta) = 0$ on $M = J^{-1}(\mu)$ and the above implies that $TJ[\Xi, H] = 0$, that is, $[\Xi, H] \in TM$. Therefore, we have $[[\Xi, H], F] = 0$ on M . Thus, the Lemma and Proposition complete the proof of our theorem. $\square$

Example. Let P be $\mathbf{R}^{2n}$ with the standard coordinates $(x^1, y^1, \dots, x^n, y^n)$. Define a vector field ξ and a 2-vector field Ξ on P by

$$\xi = \sum_i \left(x^i \frac{\partial}{\partial y^i} - y^i \frac{\partial}{\partial x^i} \right)$$

and

$$\begin{aligned} 2\Xi = & + \sum_i x^i \frac{\partial}{\partial x^i} \wedge \sum_j y^j \frac{\partial}{\partial x^j} + \sum_i x^i \frac{\partial}{\partial y^i} \wedge \sum_j y^j \frac{\partial}{\partial y^j} \\ & - \sum_i x^i \frac{\partial}{\partial x^i} \wedge \sum_j x^j \frac{\partial}{\partial y^j} - \sum_i y^i \frac{\partial}{\partial x^i} \wedge \sum_j y^j \frac{\partial}{\partial y^j} + c \sum_i \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^i}, \end{aligned}$$

where c is a constant number. Then (ξ, Ξ) defines a local Lie algebra structure on $P = \mathbf{R}^{2n}$.

(i) The abelian Lie group $\mathbf{R}$ whose Lie algebra is $\mathbf{R}\xi$ acts on (P, ξ, Ξ) . If $c \neq 0$, the action of $\mathbf{R}$ has a coadjoint equivariant momentum mapping

$$J(x^1, y^1, \dots, x^n, y^n) = \frac{1}{c} \sum_i (x^i x^i + y^i y^i).$$

But $J^{-1}(0)$ is the one point set $\{0\}$.

(ii) Let G be the connected Lie subgroup of $GL(2n, \mathbf{R})$ whose Lie algebra is generated by

$$\begin{aligned} \zeta_1 &= \begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & \zeta_2 &= \begin{bmatrix} 0 & K & 0 \\ K & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \zeta_3 &= \begin{bmatrix} K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & \zeta_4 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

where I and K are 2×2 -matrices $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then G preserves the local Lie algebra structure. If $c \neq 0$, the action of G has a coadjoint equivariant momentum mapping J defined by $\hat{J}(\zeta_K) = F_K$, where

$$\begin{aligned} F_1 &= \frac{2}{c}(-x^1 y^2 + x^2 y^1), & F_2 &= \frac{2}{c}(x^1 x^2 + y^1 y^2), \\ F_3 &= \frac{1}{c}(x^1 x^1 + y^1 y^1), & F_4 &= \frac{1}{c}(x^2 x^2 + y^2 y^2). \end{aligned}$$

It turns out that $J^{-1}(0) = \{(x^1, y^1, \dots, x^n, y^n) \in \mathbf{R}^{2n} \mid x^1 = y^1 = x^2 = y^2 = 0\} \simeq \mathbf{R}^{2n-4}$, G acts trivially on $J^{-1}(0)$ and the reduced space is of dimension $2(n-2)$ and has a local Lie algebra structure of the same type as the original local Lie algebra structure of $P = \mathbf{R}^{2n}$.

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