A CHARACTERIZATION OF THE VERONESE SURFACE

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ABSTRACT. Here we prove a slight modification of a conjecture of Beltrametti-Sommese proving that the Veronese surface and a general intersection of 3 quadrics are the only smooth surfaces of $\mathbf{CP}^5$ which are 2-spanned.

The aim of this short note is the proof of a slight modification of a conjecture raised in [BS, Conjecture 2.6], proving a slightly different and slightly more general result (see Theorem 1). The main tools will be an enumerative formula, a well-known theorem on projective curves and the classification given in [BS, §5], for embedded surfaces of sectional genus at most 5. To state the result we need to introduce a few notations. We work over the complex number field. Let S be a smooth, complete surface, $L \in \text{Pic}(S)$ and $W \subseteq H^0(S, L)$. A finite subscheme Z of S is called *curvilinear* if it is contained in a smooth curve, i.e. if for all points $P \in \text{Supp}(Z)$, Z is given around P by equations $x = y^m = 0$, x and y suitable local coordinates around P . According to [BFS] and [BS], we say that W k -spans S , k an integer ≥ 0 , if for all curvilinear subschemes Z of S with $\text{length}(Z) = k + 1$, the restriction map from W to $H^0(Z, L|_Z)$ is surjective. L is called k -spanned if $H^0(S, L)$ k -spans S . A smooth surface $S \subset \mathbf{P}^h$ is called k -spanned if the linear system on S determined by the embedding in $\mathbf{P}^h$ k -spans S . There are other, perhaps more natural, definitions of k -spannedness (see [BFS, §4]), but not only is the one given here is the one used heavily in [BFS] and [BS], but also it seems the weakest one among the natural possible definitions, and so the one with which Theorem 1 is strongest. It is easy to check that if W is $(k + 1)$ -spanned, then it is k -spanned. In [BFS, 0.4.1], it was noted that W is 1-spanned if and only if it embeds S in a projective space (this is true even for noncomplete linear systems). In [BS, 2.4], as a corollary of a more general result, it was proved that if L k -spans S for some $k \geq 2$, then $H^0(S, L) \geq 6$. In [BS, 2.6], it was conjectured that if L k -spans S for some $k \geq 2$ and $H^0(S, L) = 6$, then $S \cong \mathbf{P}^2$ and L gives the Veronese embedding. It is easy to check that the Veronese surface in $\mathbf{P}^5$ is 2-spanned; this follows also from general results

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in [BFS]: L is the tensor power of two very ample line bundles. This note contains only the proof of the following theorem.

Theorem 1. *Assume that W 2-spans the smooth, complete surface S . Then $\dim(W) \geq 6$ and if $\dim(W) = 6$, then either $S \cong \mathbf{P}^2$ and W gives the Veronese embedding into $\mathbf{P}^5$, or S is the intersection of 3 quadric hypersurfaces. In the latter case S is 2-spanned if and only if it contains no line; the set of such smooth complete intersections containing at least a line forms a non-empty hypersurface in the variety of all smooth complete intersections of 3 quadric hypersurfaces in $\mathbf{P}^5$.*

In particular note that the Veronese surface is the only smooth surface in $\mathbf{P}^5$ such that all smooth embedded deformations in $\mathbf{P}^5$ are again 2-spanned.

Note that by definition $S \subset \mathbf{P}^n$ is a 2-spanned surface, then there is no line $D \subset S$ and no line $R \not\subset S$ with $\text{length}(R \cap S) \geq 3$ (a “trisecant line”). Thus $\dim(W) > 4$. Consider S embedded in $\mathbf{P}^n$ by W . Then for any hyperplane H , $H \cap S$ contains no “trisecant line”. For general H , $S \cap H$ is a smooth, irreducible space curve (Bertini’s theorem). Set $d := \deg(S)$ and let g be the genus of $S \cap H$.

First assume $\dim(W) = 5$. Fix a point P of $S \cap H$; there is no “trisecant” line to $S \cap H$ passing through P if and only if the image of the projection of $S \cap H$ from P into $\mathbf{P}^2$ has degree $d - 1$ and is isomorphic to $S \cap H$, hence of arithmetic genus g . By the genus formula for plane curves and Castelnuovo’s bound for the genus of space curves ([ACGH, p. 116]), one sees immediately that for any point P of $S \cap H$, there is a “trisecant line” to $S \cap H$ passing through P , unless $(d, g) = (3, 0)$ or $(4, 1)$. If $d = 3$, then S is a minimal degree surface in $\mathbf{P}^4$, hence a scroll, hence it contains many lines, contradiction. Assume $(d, g) = (4, 1)$. Since $S \cap H$ is linearly normal, so is S . Note that $S \cap H$ is the intersection of two quadrics. From the exact sequence

$$(1) \quad 0 \rightarrow \mathcal{I}_S(1) \rightarrow \mathcal{I}_S(2) \rightarrow \mathcal{I}_{S \cap H, H}(2) \rightarrow 0$$

and Bezout’s theorem, one sees that S is the intersection of two quadrics. There are several ways to check that any such S contains a line, contradicting the 2-spannedness. One can work in the Grassmannian G of line in $\mathbf{P}^4$, take as X the codimension 3 ([GH, p. 739] cycle of lines contained in a fixed smooth quadric and note that $X^2 \neq 0$. Or one can use the fact that S is a Del Pezzo surface, obtained by blowing-up 5 suitable points; a line on S is the strict transform on S of the conic through these 5 points ([D] in positive characteristic). Note that this part of the theorem, i.e. the inequality $\dim(W) \geq 6$, works even if the base field has positive characteristic.

Now assume $\dim(W) = 6$. By a theorem of Le Barz ([LB, p. 182]), the cycle of trisecant lines to $S \cap H$ has degree $t(d, g) := (d-2)(d-3)(d-4)/6 - g(d-4)$; since by assumption $S \cap H$ has a finite number of trisecant lines (indeed none) this number represents the number of trisecant lines to $S \cap H$, counted with suitable multiplicities. Thus $t(d, g) = 0$. If $d = 4$, then S is a minimal degree

surface, hence either a Veronese surface or a scroll. Since a scroll contains infinitely many lines, the latter case is impossible. Thus we may assume $d > 4$. Since $t(d, g) = 0$, we find $g = (d-2)(d-3)/6$. Let $m_1, \varepsilon_1, \mu_1$ be defined by: $m_1 := [(d-1)/4]$, $\varepsilon_1 := d - 4m_1 - 1$, $\mu_1 := 1$ if $\varepsilon_1 = 3$, $\mu_1 := 0$ otherwise. Set $\pi_1(d, 4) := m_1(m_1 - 1)/2 + m_1(\varepsilon_1 + 1) + \mu_1$. First assume $g > \pi_1(d, 4)$. By [ACGH, p. 123], if $d > 10$ $S \cap H$ is contained in a minimal degree surface T of H , $\deg(T) = 3$. T is either a smooth scroll or the cone over a rational normal curve in $\mathbf{P}^3$. First assume that T is a cone. Let P be the vertex of T . Since $S \cap H$ has no “trisecant line”, it is smooth, and $g > 0$, $S \cap H$ does not contain P and intersects each line of T in two points. The minimal desingularization of T is the Hirzebruch surface F_3 . $\text{Pic}(F_3)$ has as basis h, f (f is the fiber while h is contracted to P) with the relations: $h^2 = -3$, $h \cdot f = 1$, $f^2 = 0$. Furthermore the pull-back of the hyperplane class of T is $h + 3f$. The inverse image of $S \cap H$ in F_3 must have $2h + 6f$ as its class. Thus $g = 2$, contradicting the classification (up to a case) of all 2-spanned polarized surfaces with sectional genus at most 5, given in [BS, §5]. Now assume that T is a smooth scroll. One checks easily that T is isomorphic to the Hirzebruch surface F_1 , with h and f as basis of $\text{Pic}(T)$, with the relations $h^2 = -1$, $h \cdot f = 1$, $f^2 = 0$, and with $h + 2f$ as class of the hyperplane section. Since $H \cap S$ has $g > 0$ and no “trisecant line”, the class of $H \cap S$ is represented by $2h + bf$ for some integer b . Hence $d = b + 2$. By the adjunction formula, we find $g = b - 2$, contradiction. The case $d = 10$ cannot occur, since g is an integer.

Now assume $d = 9$, hence $g = 7$. Set $m := [(d-1)/3]$, $\varepsilon := d - 1 - 3m$, $\pi(d, 4) := 3m(m-1)/2 + m\varepsilon$ ([ACGH, p. 116]). Note that $\pi(9, 4) = 7 = g$. By [ACGH, Theorem 2.5(iii) p. 122], $H \cap S$ has infinitely many trisecant lines, contradiction.

Now assume that $g \leq \pi_1(d, 4)$. Checking the 4 possible congruence classes of $d \bmod (4)$, one sees that $d \leq 8$, hence $g \leq 5$. Furthermore $d = 7$ is excluded since g is an integer. If $(d, g) = (5, 1)$ or $(6, 2)$, the hyperplane bundle of $H \cap S$ is non-special, hence $S \cap H$ and S are linearly normal. If $d = 8$, $S \cap H$ (hence S) must be linearly normal both if the hyperplane section is special or not. Thus we may apply the results in [BS, §5]. By [BS, §5], to prove Theorem 1 it is sufficient to consider the case of a smooth S , S intersection of 3 quadric hypersurfaces in $\mathbf{P}^5$. Such a surface S is not 2-spanned if and only if it contains a line, since a line R is “trisecant” to S if and only if $R \subset S$ (Bezout’s theorem). Let $G := G(1, 5)$ be the Grassmannian of lines in $\mathbf{P}^5$, $\dim(G) = 8$, and $X(Q)$ the class of the lines contained in a fixed smooth quadric Q . $X(Q)$ has codimension 3 in G ([GH, p. 739]). For every $g \in \text{Aut}(\mathbf{P}^5) = \text{Aut}(G)$, we have $g(X(Q)) = X(g(Q))$. Hence by Bertini’s theorem ([K]) the general intersection of 3 quadrics contains no line. Using Bertini’s theorem one checks easily that there are smooth S containing a line, hence not 2-spanned; for completeness’ sake, we give the details. Take

a line R and let A be the set of quadrics containing R , take Q' and Q'' general in A and set $V := Q' \cap Q''$; V is smooth; since for every $x \in R$, the set $A(x)$ of quadrics $Q \in A$ with $T_x Q = T_x V$ has codimension 2 in A , the general $Q \in A$ is not tangent to V along R ; by Bertini's theorem for general $Q \in A$, $Q \cap V$ is smooth. Any such surface S is a $K3$ surface; any line $R \subset S$ has self-intersection -2 and $x + 1$ as Hilbert polynomial; thus any such S contains at most finitely many lines. Counting dimensions, we see that the set of such S containing at least a line has codimension 1 in the set of smooth complete intersection of 3 quadrics.

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