MICROLOCAL HYPO-ANALYTICITY AND HYPO-ANALYTIC PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. Let A be a hypo-analytic pseudodifferential operator on a hypo-analytic manifold Ω of maximal dimension. It is proved here that the operator A decreases the hypo-analytic wave front set of a distribution.

1. HYPO-ANALYTIC MANIFOLD

In [1] Baouendi, Chang, and Treves introduced generalizations of real analytic structures which they called hypo-analytic structures. Let Ω be a C^∞ manifold of dimension $m + n$. A hypo-analytic structure on Ω is the data of an open covering (U_α) of Ω and for each index α , of m C^∞ functions $Z_\alpha^1, \dots, Z_\alpha^m$ satisfying the following two conditions:

- (i) $dZ_\alpha^1, \dots, dZ_\alpha^m$ are linearly independent at each point of U_α ;
- (ii) if $U_\alpha \cap U_\beta \neq \emptyset$, there are open neighborhoods O_β of $Z_\beta(U_\alpha \cap U_\beta)$ and O_α of $Z_\alpha(U_\alpha \cap U_\beta)$ and a holomorphic map F_β^α of O_α onto O_β , such that

$$Z_\beta = F_\beta^\alpha \circ Z_\alpha \quad \text{on} \quad U_\alpha \cap U_\beta.$$

We use the notation $Z_\alpha = (Z_\alpha^1, \dots, Z_\alpha^m): U_\alpha \rightarrow \mathbb{C}^m$. In what follows, we assume that $n = 0$ and we refer to Ω as a hypo-analytic manifold.

Definition 1.1. A distribution h defined on an open neighborhood of a point p_0 of Ω is called hypo-analytic at p_0 if there is a local chart (U_α, Z_α) of the above type whose domain contains p_0 and a holomorphic function $\tilde{h}$ defined on an open neighborhood of $Z_\alpha(p_0)$ in $\mathbb{C}^m$ such that $h = \tilde{h} \circ Z_\alpha$ in a neighborhood of p_0 .

By a hypo-analytic local chart we mean an $m + 1$ -tuple $(U, Z^1, \dots, Z^m)$ [abbreviated (U, Z)] consisting of an open subset U of Ω and of m hypo-analytic functions $Z^1, \dots, Z^m$ whose differentials are linearly independent at every point of U . We observe that since $\dim \Omega = m$ the mapping $Z = (Z^1, \dots, Z^m): U \rightarrow \mathbb{C}^m$ is a local diffeomorphism onto $Z(U)$.

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2. STATEMENT OF THE RESULT

We will reason in a hypo-analytic local chart (U, Z) in Ω . After shrinking U if necessary, we may assume that the mapping $Z = (Z^1, \dots, Z^m): U \rightarrow \mathbb{C}^m$ is a diffeomorphism of U onto $Z(U)$ and that U is the domain of local coordinates x_j ($1 \leq j \leq m$) all vanishing at a "central point" which will be denoted by 0. We will suppose $Z(0) = 0$ and denote by Z_x the Jacobian matrix of the Z^j with respect to the x_k . Substitution of $Z_x(0)^{-1}Z(x)$ for $Z(x)$ will permit us to assume that $Z_x(0) =$ the identity matrix. This in turn will allow us to take the real part of the Z^j ($j = 1, \dots, m$) as coordinates and write in these new coordinates $Z^j = x_j + \sqrt{-1}\phi_j(x)$, $j = 1, \dots, m$ where $\phi = (\phi_1, \dots, \phi_m)$ is real-valued whose differential at the origin is 0. Moreover, the functions Z^j are selected so that all the derivatives of order two of the ϕ_j vanish at the origin. Indeed this can be arranged by replacing each Z^j by

$$Z^j - \frac{\sqrt{-1}}{2} \sum \sum \frac{\partial^2 \phi_j}{\partial x_k \partial x_l}(0) Z^k Z^l.$$

Such modifications of the Z^j preserve hypo-analyticity. Since the derivatives up to order two of ϕ vanish at 0, after contracting U if necessary, we may assume that for all x, y in U :

$$(2.1) \quad |\phi(x) - \phi(y)| \leq |x - y|/2$$

and

$$|\phi(y)| \leq \frac{1}{2}|y|^2.$$

Definition 2.1. Let d be a real number. We denote by $\tilde{S}^d(U)$ the space of holomorphic functions $\tilde{a}(z, w, \theta)$ in a product set $\mathcal{O} \times \mathcal{O} \times \mathcal{E}$ with $\mathcal{O}$ an open neighborhood of $Z(U)$ and $\mathcal{E}$ an open cone in $\mathbb{C}_m \setminus \{0\}$ containing $R_m \setminus \{0\}$, which have the following property:

Given any compact subset K of $\mathcal{O}$ and any closed cone $\mathcal{E}' \subseteq \mathcal{E}$ whose interior contains $R_m \setminus \{0\}$, there is a constant $r > 0$ such that for all z, w in K and all θ in $\mathcal{E}'$, we have

$$|\tilde{a}(z, w, \theta)| \leq r(1 + |\theta|)^d.$$

Set $a(x, y, \theta) = \tilde{a}(Z(x), Z(y), \theta)$. For any $\varepsilon > 0$ and $u \in C_c^0(U)$ consider the linear operator $u \rightarrow A^\varepsilon u$ defined by

$$\begin{aligned} & A^\varepsilon u(x) \\ &= \left(\frac{1}{4\pi^3} \right)^{m/2} \int \int \exp(\sqrt{-1}\xi \cdot (Z(x) - Z(y)) - \varepsilon|\xi|^2) a(x, y, \xi) u(y) dZ(y) d\xi. \end{aligned}$$

Theorems 3.1 and 3.2 of [2] may be consolidated into:

Theorem 2.1. When $\varepsilon \rightarrow 0$, A^ε converges to a continuous linear operator $A: E'(u) \rightarrow D'(U)$ which maps $C_c^\infty(U)$ into $C^\infty(U)$ continuously. If u is hypo-analytic at 0 then Au is hypo-analytic at 0.

Following [2] we will call A a hypo-analytic pseudodifferential operator.

The object of this paper is to prove the following microlocal and hence, more precise version of the above theorem.

Theorem 2.2. *Suppose $u \in E'(U)$ is hypo-analytic at the covector $(0, \xi^0) \in T^*U$, $\xi^0 \neq 0$. If A is a hypo-analytic pseudodifferential operator, Au is hypo-analytic at $(0, \xi^0)$.*

In [1] the authors microlocalized hypo-analyticity by first adapting Sato's definition [5] and then showing its equivalence with the one derived from the Fourier–Bros–Iagolnitzer transform [4].

3. MICROLOCAL HYPO-ANALYTICITY AND PROOF OF THEOREM 2.2

Before proceeding to the proof of our theorem it will be convenient to recall Sato's microlocalization as adopted in [1]. They dealt with almost-analytic extensions of the map Z . Such an extension is a C^∞ mapping $Z_\#: U + \sqrt{-1}R_m \rightarrow \mathbb{C}^m$ such that $Z_\#(x) = Z(x)$ for every x in U and $(\partial/\partial \bar{z}_j)Z_\#^k$ vanishes to infinite order at $\text{Im } z = 0$ for all $j, k = 1, \dots, m$. In the sequel Γ is a nonempty, acute, and open cone in $R_m \setminus \{0\}$. Let A be any open subset of U , $\mathcal{O}$ any open set in $U + \sqrt{-1}R_m$ which contains A . We shall use the following notation:

$N_{\mathcal{O}}(A, \Gamma) = \{Z_\#(x + \sqrt{-1}v) \in C^m : x \in A, v \in \Gamma \setminus \{0\}, x + \sqrt{-1}v \in \mathcal{O}\}$, where $Z_\#$ is a given almost-analytic extension of Z . We shall denote by $\dot{N}_{\mathcal{O}}(A, \Gamma)$ the interior of the set $N_{\mathcal{O}}(A, \Gamma)$. It is easy to see that there is an open neighborhood $U_\#$ of U in $U + \sqrt{-1}R_m$ which is mapped diffeomorphically by $Z_\#$.

The following lemma and definitions from [1] will be needed.

Lemma 3.1. *Let A be an open subset of U , and $\mathcal{O}$ an open neighborhood of A in the open set $U_\#$ as above. Let A' be any relatively compact open subset of A , and Γ' any nonempty open cone in $R_m \setminus \{0\}$ whose closure is contained in Γ . Then there is an open neighborhood $\mathcal{O}'$ of A' in $U + \sqrt{-1}R_m$ such that*

$$N_{\mathcal{O}'}(A', \Gamma') \subseteq \{Z(x) + \sqrt{-1}Z_x(x)v : x + \sqrt{-1}v \in (A + \sqrt{-1}\Gamma) \cap \mathcal{O}\},$$

$$\{Z(x) + \sqrt{-1}Z_x(x)v : x + \sqrt{-1}v \in (A' + \sqrt{-1}\Gamma') \cap \mathcal{O}'\} \subseteq N_{\mathcal{O}}(A, \Gamma).$$

Definition 3.1. We denote by $B'_{\mathcal{O}}(A, \Gamma)$ the space of holomorphic functions $\tilde{f}$ in $\dot{N}_{\mathcal{O}}(A, \Gamma)$ which have the following property:

To every compact subset $\tilde{K}$ of $N_{\mathcal{O}}(A, \Gamma)$ there are an integer $k \geq 0$ and a constant $c > 0$ such that

$$|\tilde{f}(z)| \leq c(\text{dist}[z, Z(A)])^{-k} \quad \text{for all } z$$

in $\tilde{K} \cap \dot{N}_{\mathcal{O}}(A, \Gamma)$.

In what follows the neighborhood $\mathcal{O}$ of A will be such that $Z_\#$ maps $(A + \sqrt{-1}\Gamma) \cap \mathcal{O}$ onto $N_{\mathcal{O}}(A, \Gamma)$ diffeomorphically.

Let $\tilde{f}$ belong to $B'_{\mathcal{O}}(A, \Gamma)$ and set $f = \tilde{f} \circ Z_\#$. For $t > 0$ and t small, consider

$$|\sigma|^{-1} \int \int_{\sigma} f(x + \sqrt{-1}t\bar{v}) \chi(x) dx d\bar{v},$$

where $d\dot{v}$ is the measure on S^{m-1} induced by the Lebesgue measure on R^m , $|\sigma|$ is the measure of the set σ , and χ is an arbitrary C^∞ function compactly supported in A .

Definition 3.2. We define the boundary value bf of f on A by the formula

$$\int bf(x)\psi(x)dZ(x) = \lim_{t \rightarrow +0} |\sigma|^{-1} \iint_{\sigma} f(x + \sqrt{-1}t\dot{v})\psi(x)dZ(x)d\dot{v},$$

where ψ is an arbitrary element of $C_c^\infty(A)$, and σ is a relatively compact measurable subset of $\Gamma \cap S^{m-1}$.

Definition 3.3. Let $u \in \mathcal{D}'(U)$ and (x, ξ) be a point in $U \times (R_m \setminus \{0\})$. We say that u is hypo-analytic at (x, ξ) if there are an open neighborhood $V \subseteq U$ of x , an open neighborhood $\mathcal{O}$ of V in $\mathbb{C}^m$ and a finite collection of nonempty acute open cones Γ_k in $R_m \setminus \{0\}$ ($k = 1, \dots, \nu$), such that the following hold:

for every $k = 1, \dots, \nu$ and every $v \in \Gamma_k$, $\langle \xi, v \rangle < 0$;

for each $k = 1, \dots, \nu$, there is $\tilde{f}_k \in B'_\mathcal{O}(V, \Gamma_k)$ such that in V

$$u = b\tilde{f}_1 + \dots + b\tilde{f}_\nu.$$

Definition 3.3 is motivated by the fact that if u is hypo-analytic at (x, ξ) for every $\xi \neq 0$, then u is hypo-analytic at x . We will use it in the proof of our theorem.

Proof of Theorem 2.2. We first extend the map Z , that is, the map ϕ , to the whole space R^m , so that it continues to satisfy (2.1) (this might require a contraction of U); we may choose our extension so that ϕ is compactly supported in some neighborhood of the closure of U . We consider the holomorphic functions

$$\begin{aligned} & \tilde{A}^\varepsilon u(z) \\ &= \left(\frac{1}{4\pi^3} \right)^{m/2} \iint_{UR_m} \exp(\sqrt{-1}\xi \cdot (z - Z(y)) - \varepsilon|\xi|^2) a(z, Z(y), \xi) u(y) dZ(y) d\xi. \end{aligned}$$

For $\kappa > 0$ and $\xi \neq 0$, we have

$$\int_{R_m} \exp(-\kappa|\xi|[Z(s) - Z(y)]^2) dZ(s) = \left(\frac{\pi}{\kappa|\xi|} \right)^{m/2}.$$

We may therefore write $\tilde{A}^\varepsilon u(z)$ as

$$\begin{aligned} \tilde{A}^\varepsilon u(z) &= \left(\frac{\kappa}{4\pi^4} \right)^{m/2} \iiint \exp(\sqrt{-1}\xi \cdot (z - Z(y)) - \kappa|\xi|([Z(s) - Z(y)]^2 - \varepsilon|\xi|^2) \\ &\quad \cdot a(z, Z(y), \xi) \cdot |\xi|^{m/2} u(y) dZ(y) dZ(s) d\xi. \end{aligned}$$

The transpose of the inverse of Z_x will be denoted by Z_x^* .

We can change contour in the ξ -integration from R_m to the image of R_m under the map $\zeta(\xi) = Z_x^*(s)\xi + 4\sqrt{-1}B|\xi|(z - Z(y))$, where B is a constant

satisfying $|\operatorname{Im}\{Z_x(s)^{-1}(Z(x) - Z(s))\}| \leq B|x - s|^2$, z and y are sufficiently close to 0. We choose cones $(\Gamma_j)_{0 \leq j \leq \nu}$ satisfying

- (i) $R_m = \bigcup_{j=0}^{\nu} \Gamma_j$,
- (ii) $\operatorname{measure}_{j \neq k}(\Gamma_j \cap \Gamma_k) = 0$, and
- (iii) $\xi^0 \notin$ the closed convex hull of Γ_j for $j = 1, \dots, \nu$.

For each $j = 1, \dots, \nu$ we can find a strictly convex cone C_j in $R_m \setminus \{0\}$ and a constant $c_0 > 0$ such that $\forall v \in C_j$ and $\forall \theta \in \Gamma_j$, $v \cdot \xi^0 < 0$ while $v \cdot \theta \geq c_0|v||\theta|$.

For each $j = 0, 1, \dots, \nu$ and every $\varepsilon > 0$, define

$$\begin{aligned} \tilde{J}_j^\varepsilon u(z) = & \left(\frac{\kappa}{4\pi^4} \right)^{m/2} \iiint_{\xi \in \Gamma_j} \exp(\sqrt{-1}\zeta(\xi) \cdot (z - Z(y))) \\ & - \kappa \langle \zeta(\xi) \rangle (Z(s) - Z(y))^2 - \varepsilon \langle \zeta(\xi) \rangle^2 \\ & \cdot a(z, Z(y), \zeta(\xi)) \langle \zeta(\xi) \rangle^{m/2} u(y) \det \frac{\partial \zeta}{\partial \xi} dZ(y) dZ(s) d\xi \end{aligned}$$

and let $J_j^\varepsilon u(x) = \tilde{J}_j^\varepsilon u(Z(x))$. The proof of Theorem 2.1 in [2] shows that as $\varepsilon \rightarrow +0$ each $J_j^\varepsilon u$ converges in $\mathcal{D}'(U)$ to a distribution which we denote by $J_j u$. Moreover, $Au = \sum_{j=0}^{\nu} J_j u$.

The theorem will follow from establishing that $(0, \xi^0) \notin WF_{ha}(J_j u)$, $j = 0, 2, \dots, \nu$. Consider first $\tilde{J}_j^\varepsilon u(z)$ for some $j \geq 1$. We will show that the holomorphic functions $\tilde{J}_j^\varepsilon u(z)$ converge uniformly in a neighborhood of each point $Z(x) + \sqrt{-1}Z_x(x)v$ for each $x \in U$ and $v \in C_j$, v small enough. In order to estimate the integrand in $\tilde{J}_j^\varepsilon u(Z(x) + \sqrt{-1}Z_x(x)v)$ we look at the quantity

$$\begin{aligned} \mathcal{Q} = & \operatorname{Re}\{\sqrt{-1}Z_x^*(s)\xi \cdot (Z(x) + \sqrt{-1}Z_x(x)v - Z(y)) - 4B|\xi|(Z(x) \\ & + \sqrt{-1}Z_x(x)v - Z(y))^2 - \kappa \langle \zeta(\xi) \rangle (Z(s) - Z(y))^2\}, \end{aligned}$$

where $\zeta(\xi)$ here $= Z_x^*(s)\xi + 4\sqrt{-1}B|\xi|(Z(x) + \sqrt{-1}Z_x(x)v - Z(y))$.

We will estimate each of the three terms in $\mathcal{Q}$:

$$\begin{aligned} & \operatorname{Re}\{\sqrt{-1}Z_x^*(s)\xi \cdot (Z(x) + \sqrt{-1}Z_x(x)v - Z(y))\} \\ & = \operatorname{Re}\{\sqrt{-1}Z_x^*(s)\xi \cdot (Z(x) - Z(y))\} - \operatorname{Re}\{(Z_x^*(s)\xi) \cdot (Z_x(x)v)\} \\ & = \operatorname{Re}\{\sqrt{-1}Z_x^*(s)\xi \cdot (Z(x) - Z(s))\} \\ & \quad + \operatorname{Re}\{\sqrt{-1}Z_x^*(s)\xi \cdot (Z(s) - Z(y))\} - \operatorname{Re}\{Z_x^*(s)\xi \cdot Z_x(x)v\} \\ & \leq B(|x - s|^2 + |s - y|^2)|\xi| - \operatorname{Re}\{(Z_x^*(s)\xi) \cdot (Z_x(x)v)\} \\ & \leq B(|x - s|^2 + |s - y|^2)|\xi| - c_0/2|v||\xi| \quad (\text{Since } |\phi'| \text{ is small enough.}) \\ & \leq B(2|x - y|^2 + 3|s - y|^2)|\xi| - c_0/2|v||\xi|. \end{aligned}$$

In estimating the second term of $\mathcal{Q}$ we will repeatedly use (2.1) and the inequality $|ab| \leq (\delta^2/2)a^2 + (1/2\delta^2)b^2$.

$$\begin{aligned}
 & \operatorname{Re}\{-4B|\xi|(Z(x) + \sqrt{-1}Z_x(x)v - Z(y))^2\} \\
 &= B|\xi|\{-\operatorname{Re}(Z(x) - Z(y))^2 + 4\operatorname{Re}(Z_x(x)v) \cdot (Z_x(x)v) \\
 &\quad - 8\operatorname{Re}[\sqrt{-1}(Z(x) - Z(y)) \cdot (Z_x(x)v)]\} \\
 &\leq B|\xi|\{-3|x - y|^2 + 4|v|^2 - 8\operatorname{Re}[\sqrt{-1}(Z(x) - Z(y)) \cdot (Z_x(x)v)]\} \\
 &\leq B|\xi|\{-3|x - y|^2 + 4|v|^2 + 8[(x - y) \cdot \phi'(x)v + (\phi(x) - \phi(y)) \cdot v]\} \\
 &\leq B|\xi|\left\{-3|x - y|^2 + 4|v|^2\right. \\
 &\quad \left.+ 8\left(\frac{\delta^2}{2}|x - y|^2 + \frac{1}{2\delta^2}|\phi'(x)v|^2 + \frac{\delta^2}{2}|\phi(x) - \phi(y)|^2 + \frac{1}{2\delta^2}|v|^2\right)\right\} \\
 &\leq B|\xi|\left\{-3|x - y|^2 + 4|v|^2\right. \\
 &\quad \left.+ 8\left(\frac{\delta^2}{2}|x - y|^2 + \frac{1}{4\delta^2}|v|^2 + \frac{\delta^2}{8}|x - y|^2 + \frac{1}{2\delta^2}|v|^2\right)\right\} \\
 &\leq B|\xi|\left\{(-3 + 5\delta^2)|x - y|^2 + \left(4 + \frac{6}{\delta^2}\right)|v|^2\right\}.
 \end{aligned}$$

For y, x , and v sufficiently small, we get:

$$\operatorname{Re}\{-\kappa\langle\zeta(\xi)\rangle(Z(s) - Z(y))^2\} \leq \frac{-\kappa}{2}|\xi||s - y|^2.$$

These estimates add up to give the following estimation on $\mathcal{Q}$:

$$\mathcal{Q} \leq |\xi|\left\{B(5\delta^2 - 1)|x - y|^2 + B\left(4 + \frac{6}{\delta^2}\right)|v|^2 + \left(3B - \frac{\kappa}{2}\right)|s - y|^2 - \frac{Bc_0}{2}|v|\right\}.$$

We now choose $\delta = 1/\sqrt{10}$, $|v| < (1/20)c_0$ and $\kappa > 8B$. Then $\mathcal{Q} \leq -B|\xi|(\frac{1}{2}|x - y|^2 + |s - y|^2 + (c_0/4)|v|)$.

It follows that the absolute value of the integrand in $\tilde{J}_j^e u(Z(x) + \sqrt{-1}Z_x(x)v)$ is dominated by a constant multiple of

$$(1 + |\xi|)^{d+m/2}|u(y)|e^{-B(\frac{1}{2}|x - y|^2 + |s - y|^2 + (c_0/8)|v|)|\xi|}.$$

(Here by using the proof of Theorem 2.1 we are assuming that u is continuous and that the degree d of the amplitude is as small as we like.)

Moreover, if x' is close to x and v' is near v , that is, in a sufficiently small neighborhood of the point $Z(x) + \sqrt{-1}Z_x(x)v$, the corresponding integrand is dominated by a constant times

$$(1 + |\xi|)^{d+m/2}|u(y)|e^{-B(\frac{1}{2}|x - y|^2 + |s - y|^2 + (c_0/8)|v|)|\xi|}.$$

It follows that there is $\delta > 0$ for which $(\tilde{J}_j^e u)_e$ converges locally uniformly on $\{Z(x) + \sqrt{-1}Z_x(x)v : x \in U, v \in C_j, |v| < \delta\}$ to a holomorphic function $\tilde{J}_j u$. Suppose A' is a neighborhood of $0, A' \subseteq U$ and C'_j is a cone whose closure is contained in C_j . By Lemma 3.1 we can find a neighborhood $\mathcal{O}'$ of A' in $\mathbb{C}^m$ such that $\dot{N}_{\mathcal{O}'}(A', C'_j) \subseteq \{Z(x) + \sqrt{-1}Z_x(x)v : x \in U, v \in C_j, |v| < \delta\}$.

The estimate we established for $\mathcal{O}$ implies that $\tilde{J}_j u(z)$ belongs to the space $B'_{\mathcal{O}'}(A', C'_j)$. Next fix $v \in C'_j$. Since the degree of the amplitude a may be assumed to be as small as we wish, the dominated convergence theorem can be applied to conclude that

$$\lim_{t \rightarrow 0^+} \tilde{J}_j u(Z(x) + \sqrt{-1}Z_x(x)tv) = J_j u(x)$$

on x -compact sets. Therefore, $(0, \xi^0) \notin WF_{ha}(J_j u)$ for $j = 1, \dots, \nu$.

We next consider $\tilde{J}_0^e u(z)$. Since $(0, \xi^0) \notin WF_{ha} u$, in what follows we may assume without loss of generality that $u = bf$ for some $f \in B'_{\mathcal{O}}(U, C_0)$, $\xi^0 \cdot v < 0$ for $v \in C_0$. We may also assume that the cone Γ_0 was selected so that for some $c_1 > 0$, $\xi \in \Gamma_0$ and $v \in C_0$, $\xi \cdot v \leq -c_1 |\xi| |v|$.

We will write $\tilde{J}_0^e u(z) = (\kappa/4\pi^4)^{m/2} \int_{R_m} \int_{\Gamma_0} I^e(z, \xi, s) dZ(s) d\xi$ where

$$\begin{aligned} I^e(z, \xi, s) &= \left(\frac{\kappa}{\pi^4} \right)^{m/2} \int_U \exp(\sqrt{-1}Z_x^*(s)\xi \cdot (z - Z(y)) - 4B|\xi|(z - Z(y))^2 \\ &\quad - \kappa \langle \zeta(\xi) \rangle (Z(s) - Z(y))^2 - \varepsilon \langle \zeta(\xi) \rangle^2) \\ &\quad \cdot \langle \zeta(\xi) \rangle^{m/2} a(z, Z(y), \zeta(\xi)) g(y) u(y) \det \frac{\partial \zeta}{\partial \xi} dZ(y), \\ \zeta(\xi) &= Z_x^*(s)\xi + 4\sqrt{-1}B|\xi|(z - Z(y)) \quad \text{and} \quad g \in C_c^\infty(U), \quad g \equiv 1 \end{aligned}$$

near the support of u . Define

$$\begin{aligned} b(z, \omega, \xi, s) &= a(z, \omega, Z_x^*(s)\xi + 4\sqrt{-1}B|\xi|(z - \omega)) \left[\det \frac{\partial}{\partial \xi} (Z_x^*(s)\xi + 4\sqrt{-1}B|\xi|(z - \omega)) \right] \\ &\quad \cdot \langle Z_x^*(s)\xi + 4\sqrt{-1}B|\xi|(z - \omega) \rangle^{m/2} \end{aligned}$$

and for each $\varepsilon > 0$ let

$$\begin{aligned} h^\varepsilon(z^*, s) &= \exp\{\sqrt{-1}Z_x^*(s)\xi \cdot (z - z^*) - 4B|\xi|(z - z^*)^2 \\ &\quad - \kappa \langle \zeta \rangle (Z(s) - z^*)^2 - \varepsilon \langle \zeta \rangle^2\} \cdot b(z, z^*, \xi, s), \end{aligned}$$

where in the latter $\zeta = Z_x^*(s)\xi + 4\sqrt{-1}B|\xi|(z - z^*)$.

Recalling that $u = bf$, we have:

$$\begin{aligned} I^\varepsilon(z, \xi, s) &= \lim_{t \rightarrow +0} \left(\frac{\kappa}{4\pi^4} \right)^{m/2} \int h^\varepsilon(Z(y) \\ &\quad + \sqrt{-1}Z_y(y)v)f(Z(y) + \sqrt{-1}tZ_y(y)v)g(y) dZ(y) \\ &= \lim_{t \rightarrow +0}^{(v \in C_0)} \left(\frac{\kappa}{4\pi^4} \right)^{m/2} \int h^\varepsilon(Z(y) + \sqrt{-1}tZ_y(y)v)f(Z(y) \\ &\quad + \sqrt{-1}tZ_y(y)v)g(y)d(Z(y) + \sqrt{-1}tZ_y(y)v) \\ &= \lim_{t \rightarrow +0} \left(\frac{\kappa}{4\pi^4} \right)^{m/2} \int_{C_t} h^\varepsilon(z^*)f(z^*)\tilde{g}(z^*, t)dz^* \end{aligned}$$

where for t small enough, $\tilde{g}(z^*, t) = g(y)$ and C_t is the image of U under the map $y \rightarrow z^* = Z(y) + \sqrt{-1}Z_y(y)tv$.

Here we may have to shrink U and take $\delta > 0$ so that C_t is a manifold for $0 < t \leq \delta$.

We will apply the techniques employed in the proof of Theorem 2.1 of Chapter II in [1] to establish that for z near 0 $|I^\varepsilon(z, \xi, s)|$ is uniformly bounded for all $\varepsilon > 0$ by an integrable function in ξ and s space. Let $g_0 \in C_c^\infty(U)$, $g_0 = 1$ near 0, $\text{supp } g_0 \subseteq \{x: g(x) = 1\}$ and $0 \leq g_0 \leq 1$. We restrict t to belong to the interval $(0, \delta')$, $\delta' < \delta$ and select $\tilde{s} > 0$ such that $\tilde{s} + \delta' < \delta$. By Stokes' theorem,

$$\begin{aligned} &\int_{C_t} h^\varepsilon(z^*)f(z^*)g(z^*, t)dz^* \\ &= \int_U h^\varepsilon(Z(y) + \sqrt{-1}(t + sg_0(y))Z_y(y)v)f(Z(y) + \sqrt{-1}(t + \tilde{s}g_0(y))Z_y(y)v) \\ &\quad \cdot d(Z(y) + \sqrt{-1}(t + \tilde{s}g_0(y))Z_y(y)v). \end{aligned}$$

The techniques in [1] that we referred to imply that for all $\varepsilon > 0$,

$$\left| \int_{C_t} h^\varepsilon(z^*)f(z^*)g(z^*, t)dz^* \right| \leq (\text{constant}) \cdot \tilde{\kappa},$$

where

$$\begin{aligned} \tilde{\kappa} &\leq \sup |\exp(\sqrt{-1}Z_x(s)\xi \cdot (z - z^{**}) - 4B|\xi|(z - z^{**})^2 \\ &\quad - \kappa|Z_x(s)\xi + 4\sqrt{-1}B|\xi|(z - z^{**}))(Z(s) - z^{**})| \\ &\quad z^{**} = Z(y) + \sqrt{-1}(t + \tilde{s}g_0(y))Z_y(y)v. \end{aligned}$$

We note that since the degree of the amplitude, a may be assumed to be as small as we wish, the term $b(z, z^*, \xi, s)$ is bounded. In order to estimate $\tilde{\kappa}$, we look at

$$\mathcal{Q}(z) = \text{Re}\{(\sqrt{-1}Z_x(s)\xi \cdot (z - z^{**})) - 4B|\xi|(z - z^{**})^2 - \kappa|\xi|(Z(s) - z^{**})^2\}$$

which can be compared to

$$\tilde{\mathcal{Q}}(0) = \text{Re}\{-\sqrt{-1}Z_x^*(s)\xi \cdot z^{**} - 4B|\xi|(z^{**})^2 - \kappa|\xi|(Z(s) - z^{**})^2\}$$

which in turn is comparable to

$$\tilde{\mathcal{Q}} = \operatorname{Re}\{-\sqrt{-1}\xi \cdot z^{**} - 4B|\xi|(z^{**})^2 - \kappa|\xi|(Z(s) - z^{**})^2\}.$$

We will use the assumptions that $\phi(0) = 0$, $\phi'(0) = 0$, $\phi''(0) = 0$ and we may also have to shrink the neighborhood U . We begin by estimating

$$\begin{aligned} & \operatorname{Re}\{-\sqrt{-1}\xi \cdot z^{**} - 4B|\xi|(z^{**})^2\} \\ &= \xi \cdot \phi(y) + (t + \tilde{s}g_0(y))\xi \cdot v - 4B|\xi|(|y|^2 - |\phi(y)|^2) \\ & \quad + 4B(t + sg_0(y))^2|\xi| \operatorname{Re}(Z_y(y)v \cdot Z_y(y)v) \\ & \quad + 8B|\xi|(t + sg_0(y))[(y \cdot \phi'(y)v) + (\phi(y) \cdot v)] \\ & \leq (t + \tilde{s}g_0(y))\xi \cdot v - c_2|y|^2|\xi| \quad \text{for some } c_2 > 0 \\ & \leq -c_1(t + \tilde{s}g_0(y))|v||\xi| - c_2|y|^2|\xi|. \end{aligned}$$

Since $g_0(y) \equiv 1$ near $y = 0$, there is $\tilde{c}_1 > 0$ such that the last quantity $\leq -(\tilde{c}_1 + (c_2/2)|y|^2)|\xi|$.

Next we estimate

$$\begin{aligned} \operatorname{Re}((Z(s) - z^{**})^2) &\geq \frac{3}{4}|s - y|^2 - |Z_y(y)v|^2 - 2|Z(s) - Z(y)||Z_y(y)v| \\ &\geq \frac{3}{4}|s - y|^2 - |Z_y(y)v|^2 - (|Z(s) - Z(y)|^2)/8 - 8|Z_y(y)v|^2 \\ &\geq (c_3/\kappa)|s - y|^2 - 9|Z_y(y)v|^2 \quad \text{for some } c_3 > 0. \end{aligned}$$

Moreover, for $|v|$ small enough, the latter $\geq (c_3/\kappa)|s - y|^2 - \tilde{c}_1/2\kappa$. Combining these estimates we get: $\tilde{\mathcal{Q}} \leq -|\xi|(\tilde{c}_1/2 + c_3|s - y|^2 + (c_2/2)|y|^2)$, which gives the asserted bound on $I^e(z, \xi, s)$. It follows that $\tilde{J}_0^e u(z)$ converges to a holomorphic function near $z = 0$. Hence $(0, \xi^0) \notin WF_{ha}(Au)$.

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