NOTE ON ARC-LENGTH AND HARMONIC MEASURE

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ABSTRACT. In this note it is shown that the only Smirnov domain for which arc-length measure is harmonic measure is the disk. The proof depends on some facts about inner and outer functions and arc-length preserving maps.

A domain R bounded by a rectifiable Jordan curve is called *Smirnov* if the derivative of the conformal map which takes the unit disk onto R is an outer function. Examples include starlike domains and domains with analytic boundary. Smirnov tried for some time to show that all rectifiable Jordan domains were Smirnov domains. Finally Keldysh and Lavrentiev in [3] constructed a pathological counterexample. For a somewhat simpler construction see [1, 2, or 4].

In what follows let D be the unit disk, let Λ_1 denote arc-length measure, and let ω_a denote harmonic measure for R at the point a .

Theorem. If $\psi: R \dashrightarrow G$ is a conformal arc-length preserving map between two Smirnov domains R and G , the ψ is linear with $|\psi'| = 1$.

Proof. Suppose $\phi: D \dashrightarrow R$ is conformal. Then since R and G are Smirnov domains $\psi'(\phi(z))\phi'(z)$ and $\phi'(z)$ must be outer functions. Thus $\psi'(\phi(z)) = \psi'(\phi(z))\phi'(z)/\phi'(z)$ is also seen to be outer by considering the integral representation of an outer function. Let E be a Borel subset of ∂G . Then

$$\begin{aligned} \int_{\phi^{-1}(\psi^{-1}(E))} |\psi'(\phi(z))\phi'(z)| d\Lambda_1(z) &= \Lambda_1(E) \\ &= \Lambda_1(\psi^{-1}(E)) \\ &\quad \text{since } \psi \text{ preserves arc-length} \\ &= \int_{\phi^{-1}(\psi^{-1}(E))} |\phi'(z)| d\Lambda_1 \end{aligned}$$

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Since this holds for all Borel sets E we have that $|\psi'(\phi(z))| = 1$ a.e. on ∂D . Thus $\psi'(\phi(z))$ is both an inner and an outer function and therefore constant. The result follows. $\square$

The previous theorem shows that an arc-length preserving conformal map changes only the position and orientation of a Smirnov domain not its shape.

Corollary. *The only Smirnov domain such that $(1/2\pi)\Lambda_1 = \omega_a$ is the unit disk centered at a .*

Proof. Let $\omega_a = (1/2\pi)\Lambda_1$ for a Smirnov domain R and let $\phi: D \rightarrow R$ be a conformal map which takes 0 to a . Since conformal maps carry harmonic measure to harmonic measure and the harmonic measure for D at 0 is arc-length measure, ϕ is an arc-length preserving map. Thus ϕ is linear by the previous result and ϕ' is a constant of modulus one, so $\phi(D)$ is the unit disk centered at a . $\square$

Note. The hypothesis that R be Smirnov cannot be removed as examples in [1, 2, 3, and 4] show.

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