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A COMBINATORIAL CHARACTERIZATION OF $S^3 \times S^1$ AMONG CLOSED 4-MANIFOLDS

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ABSTRACT. The topological product $S^3 \times S^1$ is proved to be the unique closed connected 4-manifold of regular genus one. As a consequence, the complex projective plane CP^2 has regular genus two.

1. CRYSTALLIZATIONS

Throughout the paper we work in the piecewise-linear (PL) category in the sense of [RS]. The prefix PL will always be omitted. Manifolds are assumed to be connected and compact. As general reference for graph theory see [H, and W]. We shall use the term *graph* instead of multigraph, hence loops are forbidden but multiple edges are allowed.

An *edge-coloration* c on a graph $G = (V(G), E(G))$ is a map $c: E(G) \to C$ (where C is a finite nonempty set, called the *colour set*) such that $c(e) \neq c(f)$ for any two adjacent edges $e, f \in E(G)$. An $(n+1)$-*coloured graph* is a pair (G, c) where G is a graph, regular of degree $n+1$, and $c: E(G) \to \Delta_n = \{0, 1, \dots, n\}$ is an edge-coloration on G. If $\Gamma \subseteq \Delta_n$, then we set $G_\Gamma = (V(G), c^{-1}(\Gamma))$. If $\hat{i}$ denotes the set $\Delta_n - \{i\}$, then (G, c) is said to be *contracted* iff $G_{\hat{i}}$ is connected for each $i \in \Delta_n$. If $\Gamma = \{i, j\} \subseteq \Delta_n$ (resp. $\Gamma = \{r, s, t\} \subseteq \Delta_n$), then g_{ij} (resp. g_{rst}) represents the number of components of G_Γ. The *n-dimensional pseudocomplex* $K = K(G)$ (see [HW, p. 49], *associated with* (G, c) is defined as follows: (1) take an n-simplex $A^n(v)$ for each vertex $v \in V(G)$ and label its vertices by Δ_n; (2) if v, w are joined in G by an i-coloured edge, then identify the $(n-1)$-faces of $A^n(v)$ and of $A^n(w)$ opposite to the vertex labelled by i so that equally labelled vertices coincide. By abuse of language we call *simplexes* the balls of K, as each h-ball of K is actually isomorphic to a standard h-simplex. For each $\Gamma \subseteq \Delta_n$ with

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$\sharp\Gamma = h \leq n$, there is a bijection between the set of components of G_Γ and the set of $(n-h)$-simplexes of K whose vertices are labelled by $\Delta_n - \Gamma$. Note that, if (G, c) is contracted, then K has exactly $n+1$ vertices.

Now let A be a simplex of K. Then the *disjoined star* $\mathrm{std}(A, K)$ is defined as the disjoint union of the n-simplexes of K containing A with re-identification of the $(n-1)$-faces containing A and of their faces. The *disjoined link* is the subcomplex $\mathrm{lkd}(A, K) = \{B \in \mathrm{std}(A, K) | A \cap B = \varnothing\}$. The polyhedron $|K|$ is a closed n-manifold iff $\mathrm{lkd}(A^h, K)$ is a combinatorial $(n-h-1)$-sphere for each h-simplex $A^h \in K$. A *crystallization* of a closed n-manifold M is a contracted $(n+1)$-coloured graph (G, c) such that $|K(G)|$ is homeomorphic to M. Moreover we say that $K(G)$ is a *contracted triangulation* of M and that (G, c) *represents* M and every homeomorphic space. Each closed n-manifold can be represented by a crystallization (see [P]). For a general survey on crystallizations see [FGG]..

2. Regular genus

Following [G2], we state the definition of the regular genus for a closed n-manifold. A 2-*cell imbedding* (see [W]) $\alpha: |G| \to F$ of an $(n+1)$-coloured graph (G, c) into a closed surface F is said to be *regular* iff there is a cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$ of Δ_n such that each region of α is bounded by the image of a cycle of G with edges alternatively coloured by $\varepsilon_i, \varepsilon_{i+1}$ (i being an integer mod $n+1$). The *regular genus* $g(G)$ of G is the smallest integer h such that (G, c) regularly imbeds into a closed surface of genus h. The *regular genus* $g(M)$ of a closed n-manifold M is defined as the nonnegative integer

$$g(M) = \min\{g(G) | (G, c) \text{ is a crystallization of } M\}.$$

Given an $(n+1)$-coloured graph (G, c), call p the order of G divided by 2. If G is bipartite (resp. nonbipartite), for each cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$ of Δ_n there exists exactly one regular imbedding $\alpha: |G| \to F_\varepsilon$, where F_ε is the orientable (resp. nonorientable) closed surface with Euler characteristic

$$\chi_\varepsilon = \sum_{i \in Z_{n+1}} q_{\varepsilon_i \varepsilon_{i+1}} + (1-n)p$$

(see [G1, Proposition 19]).

All closed n-manifolds or regular genus zero are proved to be homeomorphic to the n-sphere S^n (see [FG, Main Theorem]). In the present paper, we are interested in closed 4-manifolds of regular genus one. Further results about closed 4-manifolds of regular genus greater than one will appear in a subsequent paper of the author.

3. Main results

Now we state our Main Theorem.

Main Theorem. *Let M^4 be a closed 4-manifold. Then $g(M^4) = 1$ if and only if M^4 is homeomorphic to $S^3 \times S^1$.*

Corollary. *If CP^2 is the complex projective plane, then $g(CP^2) = 2$.*

In order to prove these statements we need three lemmas and some preliminary constructions. Let (G, c) be a crystallization of a closed 4-manifold M, $K = K(G)$ the associated contracted triangulation of M, and $\{v_i / i \in \Delta_4\}$ the vertex-set of K. We may assume that v_i corresponds to $G_{\hat{i}}$ for each $i \in \Delta_4$. If $\{i, j\} = \Delta_4 - \{r, s, t\}$, then $K(i, j)$ (resp. $K(r, s, t)$) denotes the subcomplex of K generated by the vertices v_i and v_j (resp. v_r, v_s and v_t). Obviously the number of edges (resp. triangles) of $K(i, j)$ (resp. $K(r, s, t)$) equals g_{rst} (resp. g_{ij}). If $\operatorname{Sd} K$ is the first barycentric subdivision of K, let $H(i, j)$ be the largest subcomplex of $\operatorname{Sd} K$, disjoint from $\operatorname{Sd} K(i, j) \cup \operatorname{Sd} K(r, s, t)$. Then the polyhedron $|H(i, j)|$ is a closed 3-manifold $F(i, j)$ which splits M into two complementary 4-manifolds $N(i, j)$ and $N(r, s, t)$ having $F(i, j)$ as common boundary. Furthermore $N(i, j)$, $N(r, s, t)$ are regular neighbourhoods in M of $|\operatorname{Sd} K(i, j)|$, $|\operatorname{Sd} K(r, s, t)|$ respectively.

Lemma 1. *Let (G, c) be a crystallization of a closed 4-manifold M. For each triple (r, s, t) of distinct elements of Δ_4, we have*

$$2g_{rst} = g_{rs} + g_{st} + g_{tr} - p. \tag{1}$$

Proof. If $\{i, j, k\} \subseteq \Delta_4$, then $q_h(i, j)$ (resp. $q_h(i, j, k)$) denotes the number of h-simplexes of K containing v_i and v_j (resp. v_i, v_j and v_k) as their vertices. If $\{i, j\} = \Delta_4 - \{r, s, t\}$, then it is easily proved that

$$q_1(i, j) = g_{rst},$$
$$q_2(i, j) = q_2(i, j, t) + q_2(i, j, r) + q_2(i, j, s) = g_{rs} + g_{st} + g_{tr},$$
$$q_3(i, j) = 3p \qquad \text{and} \qquad q_4(i, j) = 2p.$$

Let now e be an arbitrary edge of $K(i, j)$. Then the Euler characteristic $\chi(e)$ of $\operatorname{lkd}(e, K)$ is given by $2 = \chi(e) = q_2(e) - q_3(e) + q_4(e)$, where $q_h(e)$ is the number of h-simplexes of K containing e as their face. Summation over the edges of $K(i, j)$ gives $2g_{rst} = 2q_1(i, j) = q_2(i, j) - q_3(i, j) + q_4(i, j) = g_{rs} + g_{st} + g_{tr} - p$ as claimed. □

Now we assume that (G, c) regularly imbeds into the closed orientable surface of genus $g = g(M)$ and of Euler characteristic

$$g_{01} + g_{12} + g_{23} + g_{34} + g_{40} - 3p = 2 - 2g. \tag{2}$$

Each subgraph $G_{\hat{i}}$ $(i \in \Delta_4)$ regularly imbeds into an orientable closed surface since $G_{\hat{i}}$ represents the combinatorial 3-sphere $\operatorname{lkd}(v_i, K)$. Then we can define the nonnegative integer $g_{\hat{i}}$ $(i \in \Delta_4)$ as follows (see §2)

$$g_{12} + g_{23} + g_{34} + g_{41} - 2p = 2 - 2g_{\hat{0}}\,, \tag{3}$$
$$g_{02} + g_{23} + g_{34} + g_{40} - 2p = 2 - 2g_{\hat{1}}\,, \tag{4}$$
$$g_{01} + g_{13} + g_{34} + g_{40} - 2p = 2 - 2g_{\hat{2}}\,, \tag{5}$$
$$g_{01} + g_{12} + g_{24} + g_{40} - 2p = 2 - 2g_{\hat{3}}\,, \tag{6}$$
$$g_{01} + g_{12} + g_{23} + g_{30} - 2p = 2 - 2g_{\hat{4}}. \tag{7}$$

By substituting each relation (k), $3 \leq k \leq 7$, into (2) and by using (1), we get

$$g_{14} = g_{014} + g - g_{\hat{0}}\,, \tag{8}$$
$$g_{02} = g_{012} + g - g_{\hat{1}}\,, \tag{9}$$
$$g_{13} = g_{123} + g - g_{\hat{2}}\,, \tag{10}$$
$$g_{24} = g_{234} + g - g_{\hat{3}}\,, \tag{11}$$
$$g_{03} = g_{034} + g - g_{\hat{4}}. \tag{12}$$

As a direct consequence, the inequalities $g \geq g_{\hat{i}} \geq g(G_{\hat{i}})$ hold for each color $i \in \Delta_4$.

Lemma 2. *With the above notation, we have*

$$g_{134} = 1 + g - g_{\hat{0}} - g_{\hat{2}}\,, \tag{13}$$
$$g_{124} = 1 + g - g_{\hat{0}} - g_{\hat{3}}\,, \tag{14}$$
$$g_{024} = 1 + g - g_{\hat{1}} - g_{\hat{3}}\,, \tag{15}$$
$$g_{023} = 1 + g - g_{\hat{1}} - g_{\hat{4}}\,, \tag{16}$$
$$g_{013} = 1 + g - g_{\hat{2}} - g_{\hat{4}}\,, \tag{17}$$
$$g_{012} + g_{014} + g_{034} + g_{123} + g_{234} = 4 + p + g - \sum_i g_{\hat{i}}\,, \tag{18}$$
$$g_{02} + g_{03} + g_{13} + g_{14} + g_{24} = 4 + 6g + p - 2\sum_i g_{\hat{i}}\,, \tag{19}$$
$$\chi(M) = 2 - 2g + \sum_i g_{\hat{i}}. \tag{20}$$

Proof. For each pair $(i, j) \in \{(0, 2), (0, 3), (1, 3), (1, 4), (2, 4)\}$, we get the formula $(11 + i + j)$ of the statement by simply adding the relations $(3 + i)$, $(3 + j)$ and by using (1). Summation directly gives

$$g_{134} + g_{124} + g_{024} + g_{023} + g_{013} = 5 + 5g - 2\sum_i g_{\hat{i}}. \tag{21}$$

Adding the following relations (see Lemma 1): $2g_{013} = g_{01} + g_{13} + g_{30} - p$, $2g_{024} = g_{02} + g_{24} + g_{40} - p$, $2g_{023} = g_{02} + g_{23} + g_{30} - p$, $2g_{134} = g_{13} + g_{34} + g_{41} - p$, $2g_{124} = g_{12} + g_{24} + g_{41} - p$ and making use of (2) and (21), we obtain formula (19).

Substituting formula (1) into (2) easily gives

$$g_{123}+g_{034}-g_{013}+g_{01}-p=1-g\,, \tag{22}$$
$$g_{012}+g_{234}-g_{024}+g_{04}-p=1-g\,, \tag{23}$$
$$g_{014}+g_{234}-g_{124}+g_{12}-p=1-g\,, \tag{24}$$
$$g_{034}+g_{012}-g_{023}+g_{23}-p=1-g\,, \tag{25}$$
$$g_{014}+g_{123}-g_{134}+g_{34}-p=1-g. \tag{26}$$

Adding these relations and using (2), (21) gives formula (18). Now call q_h $(h\in\Delta_4)$ the number of h-simplexes of $K=K(G)$. By construction, we have

$$q_0=5\,,\quad q_1=\sum_{r,s,t}g_{rst}\,,\quad q_2=\sum_{i,j}g_{ij}\,,\quad q_3=5p\quad\text{and}\quad q_4=2p\,.$$

Then the Euler characteristic $\chi(M)$ of $M=|K|$ is given by

$$\begin{aligned}\chi(M)&=\sum_h(-1)^h q_h=5-\sum g_{rst}+\sum g_{ij}-3p\\&=5-\left(4+p+g-\sum_i g_{\hat{i}}+5+5g-2\sum_i g_{\hat{i}}\right)\\&\quad+\left(3p+2-2g+4+6g+p-2\sum_i g_{\hat{i}}\right)-3p\\&=2-2g+\sum_i g_{\hat{i}}.\quad\square\end{aligned}$$

Lemma 3. *If $g=1$, then we have $g_{\hat{i}}=0$ for each $i\in\Delta_4$, $\beta_0=\beta_1=\beta_3=\beta_4=1$, and $\beta_2=0$, where β_k is the kth Betti number of M.*

Proof. If $g=1$, then M is orientable since the regular genus of a closed nonorientable 4-manifold is an even positive number (see [G2, Corollary 8]). By Lemma 2, it suffices to prove that the sum $R=g_{013}+g_{023}+g_{024}+g_{124}+g_{134}$ equals 10. By (21), Lemma 2 and $g_{\hat{i}}\le 1$ $(i\in\Delta_4)$, the inequalities $6\le R=10-2\sum_i g_{\hat{i}}\le 10$ hold, whence $R\in\{6,8,10\}$. Now we show that the cases $R=6$ or $R=8$ give a contradiction.

(I) If $R=6$, then we must have that one of the terms of R is 2 and the other four terms are 1 since $g_{rst}\ge 1$.

Now we first suppose $g_{134}=2$, so that $g_{013}=g_{023}=g_{024}=g_{124}=1$, $g_{\hat{0}}=g_{\hat{1}}=g_{\hat{2}}=0$ and $g_{\hat{3}}=g_{\hat{4}}=1$ by Lemma 2. Since $g_{124}=1$, $K(0,3)$ consists of exactly one edge, hence $N(0,3)$ is a 4-ball. Furthermore $K(1,4)$ and $K(2,4)$ are also formed by one edge each since $g_{023}=g_{013}=1$. Thus all triangles of $K(1,2,4)$ have two edges in common. Since $g_{034}-g_{03}=g_{\hat{4}}-g=0$, $K(1,2,4)$ consists of as many triangles as there are edges in $K(1,2)$. Therefore $K(1,2,4)$ is a cone over the 1-pseudocomplex $K(1,2)$. Then the polyhedron $|K(1,2,4)|$ is contractible, $N(1,2,4)$ is a 4-ball and M is homeomorphic to S^4. This contradicts the fact that $g(M)$ is not zero and also proves that

$g_{024} = 2$ is impossible. In the other three possible cases, i.e., $g_{013} = 2$, $g_{124} = 2$ or $g_{023} = 2$, we repeat the above arguments replacing the pair $(K(0,3), K(1,2,4))$ with $(K(1,4), K(0,2,3))$, $(K(0,2), K(1,3,4))$, and $(K(2,4), K(0,1,3))$, respectively.

(*//*) If $R = 8$, then $\sum_i g_{\hat{i}} = 1$ implies $\chi(M) = 1$ (see (20)). Since at least one of the g_{ijk}'s in R equals 1, the fundamental group $\prod_1(M)$ of M is null (see [G3]). Thus we have $\beta_0 = \beta_4 = 1$, $\beta_1 = \beta_3 = 0$, hence $\chi(M) = 2+\beta_2 \geq 2$, which is a contradiction.

Now the result $R = 10$ (hence $\sum_i g_{\hat{i}} = 0$) directly implies $\chi(M) = 0$ and $g_{\hat{i}} = 0$ for each $i \in \Delta_4$. Furthermore the equalities $\chi(M) = 0$ and $g_{013} = 2$ give $\prod_1(M) \simeq Z$ (compare again [G3]), $\beta_0 = \beta_1 = \beta_3 = \beta_4 = 1$, and $\beta_2 = 0$. This concludes the proof. □

4. Proof of main theorem

If M^4 is homeomorphic to $S^3 \times S^1$, then $g(M) = 1$ as proved in [FG, Corollary 1]. Now we prove the converse implication. Since $g_{024} = 2$ (by Lemma 3 and (15)), $K(1,3)$ consists of exactly two edges, hence $N(1,3)$ is homeomorphic to $S^1 \times B^3$, B^3 being a 3-ball. Since $g_{134} = g_{013} = 2$, $K(0,2)$, and $K(2,4)$ are also formed by exactly two edges each. Furthermore $K(0,2,4)$ has one more triangle than there are edges in $K(0,4)$ since $g_{13} = g_{123} + 1$ (by Lemma 3 and (10)).

Call A_1, A_2 the two triangles of $K(0,2,4)$ which have a common edge $e \in K(0,4)$ as their face. Then $K(0,2,4)$ collapses to the subcomplex $\overline{K} = K(0,2) \cup K(2,4) \cup \{A_1, A_2\}$. If $\partial A_1 \neq \partial A_2$, it is very easy to see that $\overline{K}$ collapses to a 1-sphere S^1, hence $N(0,2,4)$ is homeomorphic to $S^1 \times B^3$. If $\partial A_1 = \partial A_2$, then $\overline{K}$ is homotopy equivalent to $S_1^1 \vee S_2^1 \vee S_2^2$ (where $X \vee Y$ means the one-point union of X and Y), S_i^1, $i = 1, 2$, (resp. S_2^2) being a 1-sphere (resp. 2-sphere). In the latter case, we would have $\beta_2 \geq 1$ which contradicts $\beta_2 = 0$ (see Lemma 3). Therefore we have $N(1,3) \simeq N(0,2,4) \simeq S^1 \times B^3$, $\partial N(1,3) = \partial N(0,2,4) \simeq S^1 \times S^2$, whence $M = N(1,3) \cup N(0,2,4) \simeq S^3 \times S^1$ by Theorem 2 of [M].

Proof of Corollary. The genus of CP^2 is proved to be ≤ 2 (see [FGG, p. 134]) by constructing a simple crystallization of CP^2 with regular genus two. Thus the statement follows from our Main Theorem. □

Note. It is easily proved that the regular genus is subadditive, by direct construction. Thus the connected sum $(S^3 \times S^1)\#(S^3 \times S^1)$ also has regular genus two from our Main Theorem.

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