PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 4, April 1989

A SPLITTING THEOREM FOR COMPLETE MANIFOLDS WITH NONNEGATIVE CURVATURE OPERATOR

MARIA HELENA NORONHA

(Communicated by Jonathan M. Rosenberg)

ABSTRACT. It is shown that any complete, noncompact, simply connected Riemannian manifold with nonnegative curvature operator is isometric to the product of its compact soul (in the sense of Cheeger-Gromoll) and a complete manifold diffeomorphic to a Euclidean space

1. INTRODUCTION

It has been an important problem in Riemannian geometry to determine the structure of a complete, noncompact manifold M whose sectional curvatures are nonnegative. J. Cheeger and D. Gromoll in [CG] have shown that M is diffeomorphic to the total space of a vector bundle over a compact, totally geodesic submanifold, called the soul, and classified it in dimensions ≤ 3 up to isometry. These are the most significant results in this direction. In the same paper the authors left an interesting problem: "Suppose there is a point $x \in M$ such that all the sectional curvatures are positive. Is the soul of M a point, or equivalently, is M diffeomorphic to the Euclidean space $\mathbf{R}^n$?" This is known to be true for immersed hypersurfaces in Euclidean space.

In this paper we want to consider a stronger condition on such manifolds, namely the nonnegativity of the curvature operator (see definition below) and answer the Cheeger-Gromoll conjecture affirmatively in this case. This in turn implies a positive answer to the same conjecture for manifolds isometrically immersed in Euclidean space with codimension two, since it is a well-known result that in codimension two, the nonnegativity of the sectional curvatures are equivalent to the nonnegativity of the curvature operator (see [We]). Our result states

Theorem. *Let M^n be a complete noncompact, simply connected manifold with nonnegative curvature operator. Then M is isometric to the product $S^k \times \mathscr{P}^{n-k}$*

Received by the editors February 2, 1988 and, in revised form, April 4, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53C20.
Key words and phrases. Nonnegative curvature, soul.
This research was partially supported by FAPESP-BRASIL.

0002-9939/89 $1.00 + $.25 per page

where S is the k-dimensional soul of M and $\mathcal{P}^{n-k}$ is a complete manifold diffeomorphic to $\mathbf{R}^{n-k}$.

Remark. This result gives a complete topological description of this manifold since we know the possibilities for the soul S from the classification for simply connected, compact manifolds with nonnegative curvature operator which appears in [GM and CY]. Namely, S is a Riemannian product of manifolds of the following types: compact symmetric spaces, Kähler manifolds biholomorphic to complex projective spaces and manifolds homeomorphic to spheres.

Corollary. *Let M^n be a complete noncompact manifold with nonnegative curvature operator. Then M is locally isometric to a product over S. In particular, if the curvature operator is positive at some point, then M^n is diffeomorphic to $\mathbf{R}^n$.*

We want to observe that the nonnegativity of the curvature operator is equivalent to the nonnegativity of the sectional curvatures in two more cases.

(i) manifolds which can be immersed isometrically into space forms with flat normal connection,
(ii) submanifolds in which the second fundamental form satisfies the condition (4.13) in [KW].

For these cases, our theorem also gives an answer to the Cheeger-Gromoll conjecture.

Some of the arguments in this paper can also be found in G. Walschap [Wa]. As the referee has pointed out, our theorem follows from a stronger statement proved independently by M. Strake [S] and J.-W. Yim $[Y_2]$, whose preprints were received after this paper was completed. However, for the special case of nonnegative curvature operator, we present a simpler proof. The author wishes to thank the referee for pointing this out.

2. Basic results

For a Riemannian manifold M the curvature operator at $x \in M$ is the linear symmetric map

$$\rho: \bigwedge^2(T_X M) \to \bigwedge^2(T_X M)$$

characterized by

$$\langle \rho(X \wedge Y), (W \wedge Z) \rangle = \langle R(X, Y)Z, W \rangle$$

where the scalar product on the left-hand side is the induced one at the level of two-forms and R is the Riemannian tensor. Since ρ is symmetric, it makes sense to talk about the positivity and the nonnegativity of ρ.

Now suppose that M is a complete manifold with a soul denoted by S.

(2.1) **Proposition.** *If the curvature operator is nonnegative and* $\dim S \geq 2$, *then the inclusion $i: S \to M$ has flat normal bundle.*

Proof. For every $x \in M$, let us consider the normal set $\{w_i\}$ in $\bigwedge^2(T_xM)$ which diagonalizes ρ with eigenvalues λ_i. Then for X, $Y \in T_xM$ we write $X \wedge Y = \sum a_i w_i$ and therefore

$$\rho(X \wedge Y) = \sum a_i \rho(w_i) = \sum a_i \lambda_i w_i$$

with $\lambda_i \geq 0$. Notice that

(2.2) if the sectional curvature $K(X, Y) = 0$ we have $\rho(X \wedge Y) = 0$

since $0 = \langle \rho(X \wedge Y), X \wedge Y \rangle = \sum a_i^2 \lambda_i$ and $\lambda_i \geq 0$ for all i.

Now, take $x \in S$, $X, Y \in T_xS$ and $Z \in T_xS^\perp$. By Theorem 3.1 in [CG], $K(X, Z) = 0$ and $K(Y, Z) = 0$ which implies $\rho(X \wedge Z) = 0$ and $\rho(Y \wedge Z) = 0$. Using the first Bianchi identity, it is easy to see that $R(X, Y)Z = 0$. Applying this fact to the Ricci equation for the totally geodesic immersion $i: S \to M$, we have for all $X, Y \in TS$ and $Z, W \in TS^\perp$, $\langle R(X, Y)Z, W \rangle = \langle R_i^\perp(X, Y)Z, W \rangle$. But the first term is $\langle \rho(X \wedge Y), W \wedge Z \rangle$ which is zero since $R(X, Y)Z = 0$ and the conclusion follows.

Now suppose M simply connected so that the soul is simply connected. Proposition (2.1) implies that for each unit normal vector Z at x we can get, by parallel transportation, a parallel section of the flat normal bundle $\nu(S)$. This parallel section together with the proposition below will take us to the concept of the pseudo-soul.

(2.3) **Proposition** (Proposition 3.2, $[Y_1]$. *Let S be a soul of M. Then S has minimal volume in its homology class.*

This result was used by Yim in $[Y_1]$ to show that if Z is any parallel section of $\nu(S)$ then the map $\varphi_Z: S \times \mathbf{R} \to M$, given by $\varphi_Z(x, t) = \exp_x tZ(x)$ is an isometric immersion. In fact, by the Rauch Comparison Theorem [CE], $\varphi_Z(\cdot, t)$ is distance nonincreasing for small t which implies that φ_Z is an isometry since for each t, $S_t = \varphi_Z(S, t)$ is in the same homology class as S and its volume is not less than that of S. By the connectedness of $\mathbf{R}$, φ_Z is an isometric immersion for all $t \in \mathbf{R}$ and its image is isometric to a product manifold $S \times \mathbf{R}$. Actually this immersion is totally geodesic (see [2.7] below), and then for each t, $\overline{S}_t = \varphi_Z(S, t)$ is a totally geodesic manifold isometric to S. Yim has called it a pseudo-soul.

(2.4) **Proposition.** *If the curvature operator is nonnegative, S is simply connected and* $\dim S \geq 2$, *then the pseudo-soul $\overline{S}$ also has flat normal bundle.*

Proof. Let us consider the pseudo-soul $\overline{S} = \exp_S \bar{t} Z_1$, where Z_1 is a parallel section of $\nu(S)$. We can define for each $\bar{x} \in \overline{S}$ such that $\bar{x} = \exp_x tZ_1$ with $x \in S$, $Z_1(\bar{x})$ by $\gamma'(\bar{t})$ where $\gamma(t) = \exp_x tZ_1$. Then $\overline{Z}_1$ is a parallel section of $\nu(\overline{S})$ by construction. We want to prove that we can construct m linearly independent sections in $\nu(\overline{S})$ where m is the codimension of the soul. We fix $x \in S$ and if $Z_2, \dots, Z_m$ are unit orthogonal vectors to $Z_1(x)$, we define

$\overline{Z}_2, \ldots, \overline{Z}_m$ at $\overline{x}$, by parallel transportation along the geodesic γ. We claim that $\overline{Z}_2, \ldots, \overline{Z}_m$ belong to the normal space to $\overline{S}$, denoted by $T_{\overline{x}}\overline{S}^1$. In fact, consider $\overline{y} \in \overline{S}$ such that $\overline{y} = \exp_y \overline{t} Z_1$, $y \in S$, and the curves c from x to y and $\overline{c}$ from $\overline{x}$ to $\overline{y}$ respectively. Let us consider the rectangle $f: [0, a] \times [0, \overline{t}] \to M$ defined by $f(s, t) = \exp_{c(s)} t Z_1(s)$. We have

$$(\partial f/\partial s)(s, t) = \lambda X, \qquad (\partial f/\partial t)(s, t) = \mu Z_1$$

with $X(s, t)$, $Z_1(s, t)$ having unit length and $Z_1(s, \overline{t}) = \overline{Z}_1(s)$. Since the Lie bracket $[\partial f/\partial s, \partial f/\partial t] = 0$, this will be

$$\lambda X(\mu) Z_1 + \lambda\mu \nabla_X Z_1 - \mu Z_1(\lambda) X - \mu\lambda \nabla_{Z_1} X = 0. \tag{2.5}$$

We have omitted t for brevity. We see that

(i) $\nabla_X Z_1 = 0$, since Z_1 is parallel,
(ii) $\langle \nabla_{Z_1} X, Z_1 \rangle = 0$ because $\varphi_{Z_1}(S)$ is a product,
(iii) $\langle \nabla_{Z_1} X, X \rangle = 0$ because X is unitary.

Then, (i), (ii) and (iii) imply in (2.5) that $\nabla_{Z_1} X = 0$. This implies that $T_x \overline{S}_t$ is parallel along γ and then if $Z_2, \ldots, Z_m \in T_x S^\perp$, $\overline{Z}_2, \ldots, \overline{Z}_m \in T_{\overline{x}} \overline{S}^\perp$.

Now we make a parallel transportation of $\overline{Z}_2, \ldots, \overline{Z}_m$ along $\overline{c}$ and we write the expression for $R(X, \overline{Z}_1)\overline{Z}_i$, $i \geq 2$, which is zero by (2.2):

$$R(X, \overline{Z}_1)\overline{Z}_i = \nabla_X \nabla_{\overline{Z}_1} \overline{Z}_i - \nabla_{\overline{Z}_1} \nabla_X \overline{Z}_i - \nabla_{[X, \overline{Z}_1]} \overline{Z}_i = \nabla_X \nabla_{\overline{Z}_1} \overline{Z}_i = 0$$

since $\nabla_X \overline{Z}_i = 0$ and $[X, Z_1] = 0$ by (i), (ii), (iii) and (2.5). It follows that

$$\partial(\langle \nabla \overline{Z}_1(s) \overline{Z}_i(s), \nabla_{\overline{Z}_1(s)} \overline{Z}_i(s) \rangle)/\partial s = 0. \tag{2.6}$$

But $\overline{c}(0) = \overline{x}$ and $\nabla_{\overline{Z}_1(0)} \overline{Z}_i(0) = 0$. So, (2.6) implies $\nabla_{\overline{Z}_1(s)} \overline{Z}_i(s) = 0$ for each s. This means that the vectors $\overline{Z}_2, \ldots, \overline{Z}_m$ obtained along $\overline{c}$ by parallel transportation are the same vectors that we would obtain making parallel transportation of $Z_2, \ldots, Z_m$ from x to y along c and then along the geodesic $\psi(t) = \exp_y t Z_1$. Since by Proposition (2.1), the parallel transportation in S does not depend on the curve c joining x to y, the parallel transportation in $\overline{s}$ from $\overline{x}$ to $\overline{y}$ will not depend on the curve $\overline{c}$ joining $\overline{x}$ to $\overline{y}$ either. This implies the proposition.

(2.7) *Remark.* We observe that the above proof also shows that the isometric immersion φ_{Z_1} is totally geodesic. Since for each $x \in S$, $\exp_x t Z_1$ is a geodesic in M, all we need is to prove that for each t, $\overline{S}_t$ is a totally geodesic submanifold of M. Then, if $X(t)$ and $Y(t)$ are vector fields tangent to $\overline{S}_t$, we have for every i

$$\frac{d}{dt}\langle \nabla_X Y, Z_i \rangle = \langle \nabla_{Z_1} \nabla_X Y, Z_i \rangle + \langle \nabla_X Y, \nabla_{Z_1} Z_i \rangle = 0 \tag{2.8}$$

because $R(Z_1, X)Y = 0$ and $[Z_1, X] = 0$ imply that $\nabla_{Z_1} \nabla_X Y = \nabla_X \nabla_{Z_1} Y = 0$ and Z_i is parallel along γ. Since for $t = 0$ we have $\langle \nabla_X Y, Z_i \rangle = 0$ because S is totally geodesic, (2.8) implies that $\overline{S}_t$ is also totally geodesic.

(2.9) **Proposition.** *Let M be a manifold as in Proposition* (2.4). *Then there exists a smooth foliation of M by totally geodesic manifolds isometric to S.*

Proof. First, we prove that for each point $x \in M$ there exists a totally geodesic manifold $\overline{S}$ isometric to S such that $x \in \overline{S}$. For that, consider $\gamma: [0, a] \to M$ the minimal connection from x to S. $\gamma'(a) \in \nu(S)$. Let Z be the parallel normal field defined on S such that $Z(\gamma(a)) = \gamma'(a)$. Then we have a pseudo-soul $\overline{S} = \exp_S aZ$ and $x \in \overline{S}$.

We claim that there exists only one totally geodesic manifold $\overline{S}$ such that $x \in \overline{S}$ and $\overline{S}$ is isometric to S. Suppose that there exists $\widetilde{S}$ with the same conditions and $x \in \widetilde{S}$. Let $\overline{S}$ be a pseudo-soul containing x. If $X \in T_x\widetilde{S}$ and $X \notin T_x\overline{S}$, we consider $\overline{X}$ the unitary orthogonal projection of X on $T_x\overline{S}^{\perp}$. Since $\overline{S}$ has flat normal bundle we take the parallel transportation of $\overline{X}$ along $\overline{S}$. Let us call $\overline{M} = \exp_{\overline{S}} t\overline{X}$. $\overline{M}$ has $\overline{S}$ as a soul, since $\overline{M}$ is isometric to $\overline{S} \times \mathbf{R}$. The vector X belongs to $T_x\overline{M}$ and is transversal to $\overline{S}$. By Theorem (5.1) of [CG], the geodesic $\sigma(t) = \exp_x tX$ must go to infinity. Since $\overline{M}$ and $\widetilde{S}$ are totally geodesic, σ is a geodesic in M and $\widetilde{S}$ going to infinity, contradicting that $\widetilde{S}$ is compact.

This shows that the foliation is well defined. We need to prove the smoothness of the foliation. Let $\overline{S}$ be the leaf containing x. Let us take ε smaller than the injectivity radius of $\nu(\overline{S})$. Now we exponentiate the global sections of $\nu(\overline{S})$ at distances smaller than ε and we get totally geodesic manifolds isometric to $\overline{S}$ which coincide with the leaves by uniqueness.

3. Proof of the theorem

By Proposition (2.9) we have two differentiable distributions defined on M, the first one D_1, given by the tangent vectors to the leaves of the foliation F and the second $D_2 = D_1^{\perp}$. We will prove that D_1 and D_2 are involutive and parallel and the theorem will follow by Frobenius.

In order to prove this, notice that the leaves of F are equidistant and simply connected. Then we can apply the Theorem of R. Hermann in [H] which says that M/F is a smooth manifold and admits a Riemannian metric for which the projection $\Phi: M \to M/F$ is a Riemannian submersion. We see that for this submersion, horizontal vectors are orthogonal to the pseudo-souls and vertical vectors are tangent to the pseudo-souls. Now, it is easy to calculate the O'Neill tensors (see [O]). With $\mathscr{H}$ and $\mathscr{V}$ denoting the projections onto the horizontal and vertical subspaces and X and V being horizontal and vertical vectors respectively, we have

$$T_V X = \mathscr{V}(\nabla_V X) \qquad A_X V = \mathscr{H}(\nabla_X V).$$

T is zero because the pseudo-souls are totally geodesic. Then it will be enough to prove that A is zero. By the Corollary 1 of [O] we have for the sectional curvature of the plane spanned by X and V

$$K(X, V) = \langle(\nabla_X T)_V V, X\rangle + \|A_X V\|^2 - \|T_V X\|^2.$$

But $K(X, Y) = 0$ and $T_V X = 0$. Then, all we need is to prove that

$$\langle(\nabla_X T)_V V, X\rangle = \langle\nabla_X T_V V, X\rangle - \langle T_{\nabla_X V} V, X\rangle - \langle T_V \nabla_X V, X\rangle = 0.$$

In fact, using again that the pseudo-souls are totally geodesic we have

$$T_V V = \mathscr{H}(\nabla_V V) = 0,$$

$$\langle T_{\nabla_X V} V, X\rangle - \langle T_V \nabla_X V, X\rangle =$$
$$\langle \mathscr{H}(\nabla_{\mathscr{V}(\nabla_X V)} V) - \mathscr{H}(\nabla_V \mathscr{V}(\nabla_X V)) - \mathscr{V}(\nabla_V \mathscr{H}(\nabla_X V)), X\rangle = 0.$$

Hence, it follows that $A_X V = 0$.

4. Proof of the corollary

Let us consider S the soul of M and $\widetilde{S}$ and $\widetilde{M}$ the respective universal coverings. By Theorem 9.1 of [CG], $\widetilde{S}$ is isometrically diffeomorphic to $S_0 \times \mathbf{R}^m$ with S_0 compact and the splitting is in the sense of Toponogov [T]. Then these lines in $\widetilde{S}$ must split off in $\widetilde{M}$ too and hence $\widetilde{M}$ is isometrically diffeomorphic to $M_0 \times \mathbf{R}^m$. But M_0 is simply connected and by the previous theorem, $M_0 = S' \times \mathscr{P}^r$, where S' is the soul of M_0.

We claim that $S_0 = S'$. For that, consider $X \in T_x S_0$. Suppose that $X \notin T_x S'$ and take the geodesic $\sigma(t) = \exp_x tX$. This geodesic, again by Theorem 5.1 of [CG], must go to the infinity contradicting the compactness of S_0. The $S_0 \subset S'$. Since S is totally convex, $\widetilde{S}$ and S_0 are totally convex. Now we have S_0 and S' compact, totally convex and without boundary. Applying Theorem 2.1 of [CG], we see that S_0 and S' have the same homotopy type. Since $S_0 \subset S'$ and both are compact we have the claim.

Now, we have the following diagram

$$\begin{tikzcd} S_0 \times \mathscr{P}^r \times \mathbf{R}^m \arrow[r, "\Pi"] \arrow[d, "P_1"'] & M \arrow[d, "P_2"] \\ S_0 \times \mathbf{R}^m \arrow[r, "\Pi"] & S \end{tikzcd}$$

where Π is the covering map and P_1 the projection onto the first factor. Since Π is a local isometry and the fundamental group preserves the splitting $S_0 \times \mathscr{P}^r \times \mathbf{R}^m$, P_1 induces a submersion $P_2 \colon M \to S$, which is a local product.

In particular, if there is a point such that the curvature operator is positive, S must be a point and the corollary follows.

References

[CE] J. Cheeger and D. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland, Amsterdam, 1975.

[CG] J. Cheeger and D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. (2) **96** (1972), 413–443.

[CY] B. Chow and D. Yang, *A classification of compact Riemannian manifolds with nonnegative curvature operator*, preprint.

[GH] S. Gallot and D. Meyer, *Opérateur de courbure et Laplacien des formes différentielles d'une variété Riemanniene*, J. Math. Pures Appl. **54** (1975), 285–304.

[H] R. Hermann, *On the differential geometry of foliations*, Ann. of Math. (2) **72** (1960), 445–457.

[KW] N. Kleinjohann and R. Walter, *Nonnegativity of the curvature operator and isotropy for isometric immersions*, Math. Z. **181** (1982), 129–142.

[O] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469.

[S] M. Strake, *A splitting theorem for open nonnegatively curved manifolds*, (to appear in Manuscripta Math.)

[T] V. A. Toponogov, *Spaces with straight lines*, Amer. Math. Soc. Transl. **37** (1964), 287–290.

[Wa] G. Walschap, *A splitting theorem for 4-dimensional manifolds of nonnegative curvature*, Proc. Amer. Math. Soc. (to appear)

[We] A. Weinstein, *Positive curved n-manifolds in* $\mathbf{R}^{n+2}$, J. Differential Geom. **4** (1970), 1–4.

[Y_1] J.-W. Yim, *Distance nonincreasing retraction on a complete open manifold of nonnegative curvature*, Ann. Global Anal. Geom. (to appear)

[Y_2] ——, *Space of souls in a complete open manifold of nonnegative curvature*, preprint.

Departamento de Matematica-IMECC, UNICAMP-C.P. 6065, 13081-Campinas-S.P. -Brasil