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ALGEBRAS OF OPERATORS ISOMORPHIC TO THE CIRCULANT ALGEBRA

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ABSTRACT. The algebra of $n \times n$ circulant matrices has a specific structure. This paper displays different operators on linear vector spaces that have the same structure, i.e. are isomorphic.

1. INTRODUCTION

Complex $n \times n$ circulant matrices are a matrix representation of the group ring (over $\mathbf{C}$) of the cyclic group. P. J. Davis [1] also proves that the set of circulants with complex entries have an idempotent basis. This paper displays algebras of operators which are isomorphic to the algebra of $n \times n$ complex circulant matrices.

§2 reviews properties of circulants and introduces a cyclic group of automorphisms on the circulant algebra generalizing conjugation. The group ring over $\mathbf{C}$ of this group is isomorphic to that of circulants themselves.

In §3, functional equations, whose solutions are functions $\mathbf{C}^n \to \mathbf{C}$, are solved using cyclic and idempotent linear operators on the space (labeled U) of functions $\mathbf{C}^n \to \mathbf{C}$. Again, this algebra of linear operators is isomorphic to $n \times n$ circulants.

§4 displays cyclic and idempotent linear operators on the space V of functions on $n \times n$ complex circulants. Furthermore, §4 shows a relationship between the operators on V and those on U.

Finally, §5 shows a linear involution on V whose group ring is isomorphic to 2×2 complex circulant matrices.

T. Muir in his classical book on determinants (cf. [3], 1920) discussed properties of circulant matrices. K. B. Leisenring in the years 1969–1979 lectured extensively on the bicomplex plane employing 2×2 circulant matrices (see his ms. book [4]). The work of Davis, Muir, and Leisenring influenced the author in various ways. Already in 1971, A. C. Wilde [2] discussed aspects of functional equations obtaining generalizations of odd and even functions in terms of nth roots of unity in $\mathbf{C}$. A. C. Wilde in [5, 6, 7] generalized properties of

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2×2 circulant matrices and 2-dimensional complex analysis to $n \times n$ circulant matrices. More work continuing the present paper is forthcoming.

2. Properties of circulants

An $n \times n$ circulant matrix is a square matrix like the following:

$$X = \begin{bmatrix} x_0 & x_1 & x_2 \cdots x_{n-1} \\ x_{n-1} & x_0 & x_1 \cdots x_{n-2} \\ x_{n-2} & x_{n-1} & x_0 \cdots x_{n-3} \\ \vdots & \vdots & \vdots \quad \vdots \\ x_1 & x_2 & x_3 \cdots x_0 \end{bmatrix}. \tag{1}$$

Let A_n denote the set of circulant matrices with complex entries, or $F = \mathbf{C}$. Let K denote the circulant matrix with $x_1 = 1$ and $x_j = 0$ for $j \neq 1$. Then K^h (the hth power of K, $1 \le h \le n$) is the circulant matrix with $x_h = 1$ and $x_j = 0$ for all $j \neq h$, $K^0 = I$ (the identity matrix), $K^1 = K$, and $K^n = I$. So X can be written as

$$X = \sum_{h=0}^{n-1} x_h K^h \tag{2}$$

for $x_0, x_1, \dots, x_{n-1} \in \mathbf{C}$. In other words, $K^0 = I$, $K, \dots, K^{n-1}$ forms a basis for the circulants A_n. Let a denote any of the nth roots of one, or $a = e^{2\pi i/n}$. Let

$$y_h = \sum_{j=0}^{n-1} a^{hj} x_j \quad \text{for } h = 0, 1, \dots, n-1. \tag{3}$$

Then, as well known (see [1]), the numbers $y_0, y_1, \dots, y_{n-1}$ are the eigenvalues of the circulant matrix X, each y_h having the corresponding eigenvector $\operatorname{Col}(1, a^h, a^{2h}, \dots, a^{(n-1)h})$.

Circulant matrices have also another basis $E_0, E_1, \dots, E_{n-1}$ defined by

$$E_h = \frac{1}{n} \sum_{j=0}^{n-1} a^{-hj} K^j \quad \text{for } h = 0, 1, \dots, n-1. \tag{4}$$

As shown in [1], these matrices have the following properties:

$$E_h^2 = E_h \quad \text{for } h = 0, 1, \dots, n-1; \tag{5.1}$$

$$E_h E_i = 0 \quad \text{for } h \neq i \quad \text{and} \tag{5.2}$$

$$E_0 + E_1 + \cdots + E_{n-1} = I. \tag{5.3}$$

Also,

$$K^h = \sum_{j=0}^{n-1} a^{hj} E_j \quad \text{for } h = 1, 2, \dots, n-1. \tag{5.4}$$

Thus, the idempotents $E_0, E_1, \dots, E_{n-1}$ form a basis for A_n, and it is shown in [1] that for X in equation (2) we also have

$$X = \sum_{h=0}^{n-1} y_h E_h = y_0 E_0 + y_1 E_1 + \cdots + y_{n-1} E_{n-1}. \tag{6}$$

We have seen that every circulant, or $X \in A_n$, can be written in one and only one way in the form (2), or

$$X = x_0 I + x_1 K + x_2 K^2 + \cdots + x_{n-1} K^{n-1}.$$

Let us now define the function $\theta\colon A_n \to A_n$ by taking (see Wilde [5])

$$\begin{aligned} \theta(X) &= x_0 I + x_1(aK) + x_2(aK)^2 + \cdots + x_{n-1}(aK)^{n-1} \\ &= x_0 I + a x_1 K + a^2 x_2 K^2 + \cdots + a^{n-1} x_{n-1} K^{n-1}, \end{aligned} \tag{7}$$

i.e., θ replaces K by aK in (2), and by composition

$$\theta^k(X) = x_0 I + a^k x_1 K + a^{2k} x_2 K^2 + \cdots + a^{(n-1)k} x_{n-1} K^{n-1}, \tag{8}$$

i.e., θ^k replaces x_h by $a^{hk} x_h$ for $h = 0, 1, \dots, n-1$. Also, $\theta^n(X) = X$. Then θ is an automorphism in A_n that preserves $\mathbf{C}$, $\mathbf{C}$ being embedded in A_n by the correspondence $z \to zI$ for $z \in \mathbf{C}$. We have seen that, for any circulant $X \in A_n$, or $X = x_0 I + x_1 K + \cdots + x_{n-1} K^{n-1}$, $x_0, x_1, \dots, x_{n-1} \in \mathbf{C}$, if we take the numbers $y_h = \sum_{j=0}^{n-1} a^{hj} x_j$, $h = 0, 1, \dots, n-1$, then relation (6) holds, or $X = y_0 E_0 + y_1 E_1 + \cdots + y_{n-1} E_{n-1}$, and then

$$\theta(X) = y_1 E_0 + y_2 E_1 + \cdots + y_{n-1} E_{n-2} + y_0 E_{n-1}, \tag{9}$$

i.e., θ shifts the eigenvalues over one space.

To generalize $\operatorname{Re} z$ and $i \operatorname{Im} z$, let $q_0, q_1, \dots, q_{n-1}$ be the functions $A_n \to A_n$ defined by

$$q_h = \frac{1}{n} \sum_{j=0}^{n-1} a^{-hj} \theta^j \quad \text{for } h = 0, 1, \dots, n-1. \tag{10}$$

Then

$$q_h^2 = q_h \quad \text{for } h = 0, 1, \dots, n-1; \tag{10.1}$$

$$q_h q_j = 0 \quad \text{for } h \neq j; \tag{10.2}$$

$$q_0 + q_1 + \cdots + q_{n-1} = \theta^0; \quad \text{and} \tag{10.3}$$

$$q_0 + a^h q_1 + a^{2h} q_2 + \cdots + a^{(n-1)h} q_{n-1} = \theta^h. \tag{10.4}$$

Also,

$$q_h(x_0 I + x_1 K + \cdots + x_{n-1} K^{n-1}) = x_h K^h \quad \text{for } h = 0, 1, \dots, n-1. \tag{11}$$

Equations (10), (10.1)–(10.3), and (11) are proved in Wilde [5]. The algebra generated by $I, K, K^2, \dots, K^{n-1}$ and $\theta^0, \theta^1, \dots, \theta^{n-1}$ over $\mathbf{C}$ are isomorphic and can be called circulant algebras.

If f is an entire function $\mathbf{C} \to \mathbf{C}$, then

$$\begin{aligned} &f(z_0 I + z_1 K + \cdots + z_{n-1} K^{n-1}) \\ &= \sum_{h=0}^{n-1} \left[\frac{1}{n} \sum_{k=0}^{n-1} a^{-hk} f\left(\sum_{j=0}^{n-1} a^{jk} z_j \right) \right] K^h \end{aligned} \tag{12}$$

for all $z_0, z_1, \dots, z_{n-1} \in \mathbf{C}$ as proved by Wilde [6].

3. Functional equations

For any entire function $f: \mathbf{C} \to \mathbf{C}$, equation (12) can be written as follows:

$$f(z_0 I + z_1 K + \cdots + z_{n-1} K^{n-1}) = \sum_{h=0}^{n-1} F_h(z_0, z_1, \dots, z_{n-1}) K^h \tag{13}$$

where

$$F_h(z_0, z_1, \dots, z_{n-1}) = \frac{1}{n} \sum_{k=0}^{n-1} a^{-hk} f\left(\sum_{j=0}^{n-1} a^{jk} z_j \right), \tag{14}$$

$h = 0, 1, \dots, n-1$. The reader may also verify that for each h, $F_0, F_1, \dots, F_{n-1}$ satisfy the functional equation

$$F(z_0, az_1, a^2 z_2, \dots, a^{n-1} z_{n-1}) = a^h F(z_0, z_1, \dots, z_{n-1}) \tag{15}$$

for $a = e^{2\pi i/n}$, all $h = 0, 1, \dots, n-1$, and all $z_0, z_1, \dots, z_{n-1} \in \mathbf{C}$.

Equation (15) is related to the circulant algebra also in another way. Let $U = \{F | F: \mathbf{C}^n \to \mathbf{C}\}$ and let C be the operator $C: U \to U$, linear in F, defined by

$$C(F)(z_0, z_1, \dots, z_{n-1}) = F(z_0, az_1, a^2 z_2, \dots, a^{n-1} z_{n-1}), \tag{16}$$

i.e. C assigns to each function $F(z_0, \dots, z_{n-1})$ in U the function $F(z_0, az_1, \dots, a^{n-1} z_{n-1})$ obtained by substituting $a^j z_j$ for z_j, $j = 0, 1, \dots, n-1$. If we denote C^k the operation C composed with itself k times, then

$$C^k(F)(z_0, z_1, \dots, z_{n-1}) = F(z_0, a^k z_1, a^{2k} z_2, \dots, a^{(n-1)k} z_{n-1}). \tag{17}$$

By Wilde [2], $C^j = C^0$ if and only if n divides j; and linear combinations of $C^0, C^1, C^2, \dots, C^{n-1}$ over $\mathbf{C}$ form a circulant algebra. Equation (15) can now be written in the form

$$C(F) = a^h F. \tag{18}$$

Also, we may define the operators $M_0, M_1, \dots, M_{n-1}: U \to U$ by taking

$$M_h = \frac{1}{n}(C^0 + a^{-h} C^1 + a^{-2h} C^2 + \cdots + a^{-(n-1)h} C^{n-1}) \tag{19}$$

for $h=0,1,\ldots,n-1$. These operators have the following properties:

$$M_h^2 = M_h \quad \text{for } h=0,1,\ldots,n-1; \tag{20.1}$$

$$M_hM_j = 0 \quad \text{if } h \neq j; \tag{20.2}$$

$$M_0 + M_1 + \cdots + M_{n-1} = C^0; \quad \text{and} \tag{20.3}$$

$$M_0 + a^hM_1 + a^{2h}M_2 + \cdots + a^{(n-1)h}M_{n-1} = C^h \tag{20.4}$$

for $h=0,1,\ldots,n-1$. (Properties (20.1)–(20.3) are proved in Wilde [7]. Properties (20.1)–(20.4) are similar to those of the functions $E_0, E_1, \ldots, E_{n-1}$ and operators $q_0, q_1, \ldots, q_{n-1}$ in §2.

By Wilde [7], a function F in U satisfies equation (15) (or (18)) if and only if $F \in \operatorname{Ran} M_h$. Moreover, properties (20.1)–(20.3) above imply that $U = \operatorname{Ran} M_0 \oplus \operatorname{Ran} M_1 \oplus \cdots \oplus \operatorname{Ran} M_{n-1}$ (a direct sum), as proved by Wilde in [7]. Each function F_h, $h=0,1,\ldots,n-1$, defined by equation (14) is in $\operatorname{Ran} M_h$ and in addition

$$F_0 + F_1 + \cdots + F_{n-1} = f(z_0 + z_1 + \cdots + z_{n-1}). \tag{21}$$

Thus

$$F_h = M_h(f(z_0 + z_1 + \cdots + z_{n-1})). \tag{22}$$

4. Other circulant algebras

By equations (19) and (17), if a function g maps $\mathbf{C}^n$ into $\mathbf{C}$, then

$$\begin{aligned} &M_h(g)(z_0, z_1, \ldots, z_{n-1}) \\ &= \frac{1}{n}\sum_{k=0}^{n-1} a^{-hk} g(z_0, a^k z_1, a^{2k} z_2, \ldots, a^{(n-1)k} z_{n-1}) \end{aligned} \tag{23}$$

for $h=0,1,\ldots,n-1$.

For $f\colon A_n \to A_n$, there exist functions $f_0, f_1, \ldots, f_{n-1}\colon \mathbf{C}^n \to \mathbf{C}$ such that

$$f\left(\sum_{h=0}^{n-1} z_hK^h\right) = \sum_{h=0}^{n-1} f_h(z_0, z_1, \ldots, z_{n-1})K^h. \tag{24}$$

Hence, from equation (20.3),

$$\begin{aligned} f\left(\sum_{h=0}^{n-1} z_hK^h\right) &= \sum_{h=0}^{n-1}\sum_{i=0}^{n-1} M_{h+i}(f_h)K^h \\ &= \sum_{i=0}^{n-1}\sum_{h=0}^{n-1} M_{h+i}(f_h)K^h. \end{aligned} \tag{25}$$

Let $V = \{f | f\colon A_n \to A_n\}$. For $f \in V$, let us define $p_i(f)$ and g_i such that

$$p_i(f)\left(\sum_{h=0}^{n-1} z_hK^h\right) = \sum_{h=0}^{n-1} M_{h+i}(f_h)K^h \tag{26}$$

and

$$g_i = \sum_{h=0}^{n-1} M_{h+i}(f_h)\,, \tag{27}$$

for $i = 0, 1, \ldots, n-1$ and $h+i$ taken modulo n. By equations (20.1) and (20.2),

$$M_{h+i}(g_i) = M_{h+i}(f_h) \tag{28}$$

for $h = 0, 1, \ldots, n-1$; $i = 0, 1, \ldots, n-1$; and $h+i$ taken modulo n. Substituting (28) into (26) yields

$$p_i(f)\left(\sum_{h=0}^{n-1} z_h K^h\right) = \sum_{h=0}^{n-1} M_{h+i}(g_i)K^h\,, \tag{29}$$

for $i = 0, 1, \ldots, n-1$ and $h+i$ taken modulo n.

Using equation (26), we can prove

$$p_i^2 = p_i \quad \text{for } i = 0, 1, \ldots, n-1; \tag{30.1}$$

$$p_i p_j = 0 \quad \text{if } i \neq j; \tag{30.2}$$

$$(p_0 + p_1 + \cdots + p_{n-1})(f) = f\,, \qquad f \in V\,, \tag{30.3}$$

i.e., $p_0, p_1, \ldots, p_{n-1}$ are orthogonal projections on V, adding to the identity function on V, and so generating over $\mathbf{C}$ a circulant algebra.

Also, we derive another formula for the projections $p_i(f)$. By equations (26), (23), (8), and (24),

$$\begin{aligned} p_i(f)\left(\sum_{h=0}^{n-1} z_h K^h\right) &= \sum_{h=0}^{n-1} M_{h+i}(f_h)K^h \\ &= \sum_{h=0}^{n-1}\left[\frac{1}{n}\sum_{k=0}^{n-1} a^{-(i+h)k} f_h(z_0, a^k z_1, \ldots, a^{(n-1)k} z_{n-1})\right] K^h \\ &= \frac{1}{n}\sum_{k=0}^{n-1} a^{-ik}\theta^{-k} f\left(\sum_{h=0}^{n-1} a^{hk} z_h K^h\right) \\ &= \frac{1}{n}\sum_{k=0}^{n-1} a^{-ik}\theta^{-k} f\theta^k\left(\sum_{h=0}^{n-1} z_h K^h\right), \end{aligned} \tag{31}$$

since θ is a function $A_n \to A_n$, namely one-to-one and onto. This result can be rewritten in the form

$$p_i(f) = \frac{1}{n}\sum_{k=0}^{n-1} a^{-ik}\theta^{-k} f\theta^k\,, \tag{31$'$}$$

for all functions $f \in V$. Finally, it can be shown that $f\theta = a^i\theta f$ if and only if $f \in \operatorname{Ran} p_i$.

Now we want to show that for each $f \in V$ and every $i = 0, 1, \ldots, n-1$, there exists only one function g_i such that equation (29) holds. Indeed, suppose there exists another function $g_i^*: \mathbf{C}^n \to \mathbf{C}$ such that $p_i(f) = \sum_{h=0}^{n-1} M_{h+i}(g_i)K^h = \sum_{h=0}^{n-1} M_{h+i}(g_i^*)K^h$. Since $I, K, K^2, \ldots, K^{n-1}$ is a basis of A_n, $M_{h+i}(g_i) = M_{h+i}(g_i^*)$ for $h = 0, 1, \ldots, n-1$. By equation (20.3),

$$g_i = \sum_{h=0}^{n-1} M_{h+i}(g_i) = \sum_{h=0}^{n-1} M_{h+i}(g_i^*) = g_i^*.$$

So g_i is unique. Thus, there is an isomorphism between functions $g_i: \mathbf{C}^n \to \mathbf{C}$ and functions $\sum_{h=0}^{n-1} M_{h+i}(g_i)K^h$ in $\operatorname{Ran} p_i$.

A result of all this is as follows: let $W = \{f \mid f: \mathbf{C} \to \mathbf{C}$ with f an entire function$\}$ and $U = \{f \mid f: \mathbf{C}^n \to \mathbf{C}\}$. Let I be a monomorphism $W \to U^n$ defined by $I(f) = (f(z_0 + \cdots + z_{n-1}), 0, \ldots, 0)$. Let $\overline{\psi}$ be a monomorphism $W \to V$ defined by

$$\overline{\psi}(f)\left(\sum_{h=0}^{n-1} z_h K^h\right) = \sum_{h=0}^{n-1} M_h(f(z_0 + \cdots + z_{n-1}))K^h = f\left(\sum_{h=0}^{n-1} z_h K^h\right),$$

which follows from equations (13), (14), and (22). Then there exists an isomorphism $\psi: U^n \to V$ defined by

$$\psi(g_0, \ldots, g_{n-1})\left(\sum_{h=0}^{n-1} z_h K^h\right) = \sum_{i=0}^{n-1}\sum_{h=0}^{n-1} M_{h+i}(g_i)K^h$$

such that the following diagram commutes:

$$\begin{tikzcd} W \arrow[d, "I"'] \arrow[dr, "\overline{\psi}"] & \\ U^n \arrow[r, "\psi"'] & V \end{tikzcd}$$

5. A linear involution

Suppose g is a function $A_n \to A_n$ given by

$$g = \sum_{i=0}^{n-1} g_i K^i \tag{32}$$

where g_i is given by equation (27). Written out, we have

$$g = \sum_{i=0}^{n-1}\left[\sum_{h=0}^{n-1} M_{h+i}(f_h)\right] K^i. \tag{33}$$

Let V denote the space of functions $A_n \to A_n$. If f is an element of V, then there exist a set of n functions $f_0, f_1, \ldots, f_{n-1}$ mapping $\mathbf{C}^n$ into $\mathbf{C}$ such that $f = \sum_{h=0}^{n-1} f_h K^h$ (like equation (24)). Let us switch the h and the i in the right-hand side of equation (33). Then let φ be the function $V \to V$ defined by

$$\varphi\left(\sum_{h=0}^{n-1} f_h K^h\right) = \sum_{h=0}^{n-1}\left(\sum_{i=0}^{n-1} M_{h+i}(f_i)\right) K^h. \tag{34}$$

We use equations (20.1)–(20.3) to show that

$$\begin{aligned}\varphi\left(\varphi\left(\sum_{h=0}^{n-1} f_h K^h\right)\right) &= \sum_{h=0}^{n-1}\left\{\sum_{i=0}^{n-1} M_{h+i}\left[\sum_{k=0}^{n-1} M_{i+k}(f_k)\right]\right\} K^h \\ &= \sum_{h=0}^{n-1}\left[\sum_{i=0}^{n-1} M_{h+i}(f_h)\right] K^h \\ &= \sum_{h=0}^{n-1} f_h K^h,\end{aligned}$$

i.e., $\varphi^2 = \varphi^0$ (the identity function on V). Thus φ is a linear involution on V. Consider the set $B = \{a_0\varphi^0 + a_1\varphi | a_0, a_1 \in \mathbf{C}\}$, i.e., linear combinations over $\mathbf{C}$ of φ^0 and φ (since $\varphi^2 = \varphi^0$). Then B is a 2×2 complex circulant algebra; $(\varphi^0 + \varphi)/2$ and $(\varphi^0 - \varphi)/2$ are idempotent elements of B, i.e., they are projections on V. If f is in V, then $\varphi(f) = f$ if and only if $f \in \operatorname{Ran}(\varphi^0 + \varphi)/2$; and $\varphi(f) = -f$ if and only if $f \in \operatorname{Ran}(\varphi^0 - \varphi)/2$. Also, $V = \operatorname{Ran}(\varphi^0 + \varphi)/2 \oplus \operatorname{Ran}(\varphi^0 - \varphi)/2$ (a direct sum).

If $n = 2$, then $K^2 = I$. Let f and g be two functions $\mathbf{C}^2 \to \mathbf{C}$. Then

$$\begin{aligned}&\varphi[If(z_0, z_1) + Kg(z_0, z_1)] \\ &\quad = I[f(z_0, z_1) + f(z_0, -z_1) + g(z_0, z_1) - g(z_0, -z_1)]/2 \\ &\qquad + K[f(z_0, z_1) - f(z_0, -z_1) + g(z_0, z_1) + g(z_0, -z_1)]/2.\end{aligned}$$

Note that, if f_i is a function $\mathbf{C}^n \to \mathbf{C}$ for each i, we have by equation (34) that

$$\varphi(f_i K^i) = \sum_{h=0}^{n-1} M_{h+i}(f_i) K^h. \tag{35}$$

Indeed, since (by equation (11))

$$q_i\left(\sum_{h=0}^{n-1} f_h K^h\right) = f_i K^i, \quad \varphi \text{ is an isomorphism}$$
$$q_i(V) \to p_i(V) \quad \text{for each } i.$$

Equation (34) implies that

$$\varphi[M_{h+i}(f_h) K^h] = M_{h+i}(f_h) K^i \tag{36}$$

where h and i vary from 0 to $n-1$ and $h+i$ is taken modulo n. Since $\varphi(f)=f$ if and only if $f \in \operatorname{Ran}(\varphi^0+\varphi)/2$, the function

$$(1/2)(\varphi^0+\varphi)[M_{h+i}(f_h)K^h] = (1/2)M_{h+i}(f_h)(K^h+K^i)$$

is a fixed point of φ.

BIBLIOGRAPHY

1. Philip J. Davis, *Circulant matrices*, Wiley-Interscience, New York, 1979.
2. Alan C. Wilde, *Solutions of equations containing primitive roots of unity*, J. Undergraduate Math., **3** (1971), 25–28.
3. Thomas Muir, *Theory of determinants*, Vol. III, (A 4 vol. treatise; see e.g. the Dover ed. 1960, and Longman ed. 1933).
4. Kenneth R. Leisenring, *The bicomplex plane* (*U-M* manuscript, submitted for publication).
5. Alan C. Wilde, *Cauchy-Riemann conditions for algebras isomorphic to the circulant algebra*, J. Univ. of Kuwait (Science), **14** (1987), 189–204.
6. ——, *Differential equations involving circulant matrices*, Rocky Mountain J. Math. **13** (1983), 1–13.
7. ——, *Commutative projection operators*, Atti del Seminario Matematico e Fisico dell'Università di Modena, **35** (1987), 167–172.

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