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A SIMPLE PROOF OF THE UNIQUENESS OF PERIODIC ORBITS IN THE 1 : 3 RESONANCE PROBLEM

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ABSTRACT. In 1979, E. Horozov considered the versal deformation of a planar vector field which is invariant under a rotation through an angle $2\pi/3$ (with resonance of order 3). In his study, the most difficult part of the proof is on the uniqueness of limit cycles. In this note we give a simple and elementary (without the theory of algebraic geometry proof of the uniqueness of periodic orbits in the 1:3 resonance problem.

In [6], Horozov considered the versal deformation of a planar vector field which is invariant under a rotation through an angle $2\pi/3$ (with resonance of order 3). The main difficulty in [6] is to prove the uniqueness of limit cycles. In this note we give a simpler proof of the uniqueness result by using the Picard-Fuchs equation and a specific technique introduced by Carr, Chow, and Hale [2] (see Proposition 4). Our proof is elementary and does not need results from algebraic geometry (see [6]).

Consider a family of vector fields with resonance of order 3. It is well known that a normal form equation of order 3 is given by the following equation (see, for example, Arnold [1] p. 293 or [7]):

$$\dot{z} = \varepsilon z + Az|z|^2 + \bar{z}^2 , \tag{1}$$

where $z, A \in \mathbb{C}$, and $\varepsilon = \varepsilon_1 + i\varepsilon_2$ is a complex parameter. For $\varepsilon = 0$, (1) becomes

$$\dot{z} = Az|z|^2 + \bar{z}^2. \tag{2}$$

The following could be found in [6].

Theorem A. *If* $\operatorname{Re} A = a \neq 0$, *then*

(a) (1) *is a versal deformation of* (2) *with resonance of order* 3;

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(b) *the bifurcation diagram of* (1) *consists of the origin and following curves in parameter space* (*Figure* 1)*:*

$$H^{\pm} = \{\varepsilon | \varepsilon_1 = 0, \varepsilon_2 \neq 0\},$$

$$HL^{\pm} = \{\varepsilon | \varepsilon_1 = -\frac{a}{8}\varepsilon_2^2 + 0(|\varepsilon_2|^3), \varepsilon_2 \neq 0\};$$

(c) *the phase portraits of* (1) *for* ε *in various regions in parameter space are shown in Figure* 1.

Remarks. (1) In Theorem A, H and HL are curves along which Hopf bifurcation and heteroclinic loop bifurcation occur. (2) In [6] there is a misprint in the equation for HL (see [6, p. 187]).

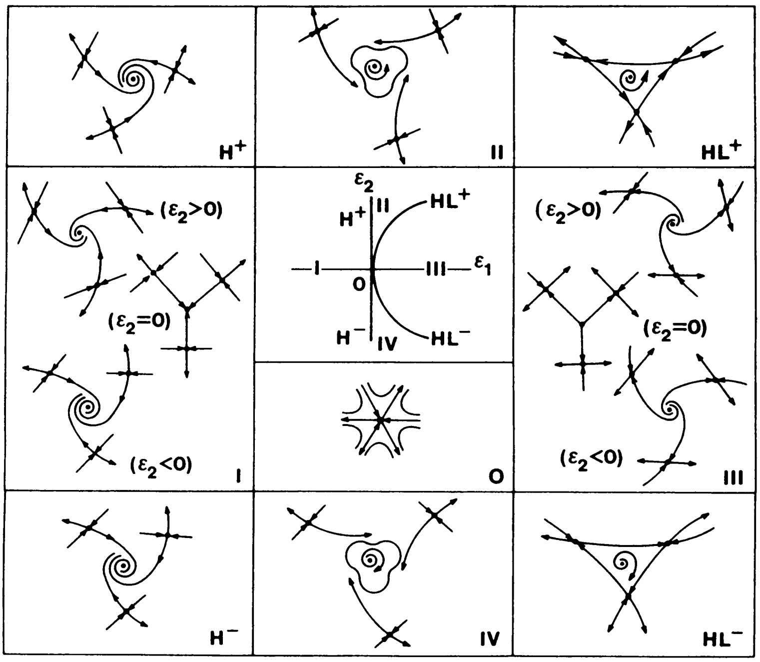

FIGURE 1.

We note that limit cycles appear only in regions II and IV in Figure 1. The difficulty in proving Theorem A is to verify the uniqueness of limit cycles in these regions. In the following, we give a proof of the uniqueness result. For a complete proof of Theorem A, we refer the reader to Horozov [6] or Chow, Li, and Wang [3].

In order to prove that equation (1) is a versal deformation of equation (2), Horozov [6] considered a planar system with Z_3-symmetry in a small neighborhood of the origin $z = 0$:

$$\dot{z} = \mu z + A(\mu) z|z|^2 + \bar{z}^2 + O(|z|^4), \tag{3}$$

where $\mu = \mu_1 + i\mu_2$ is a complex parameter. It is shown in [6] that the bifurcation diagram and phase portraits are independent of the higher order terms $O(|z|^4)$ in equation (3). It is not difficult to show that equation (3) has four equilibria in a small neighborhood of phase space. Furthermore, one of the equilibria, $z = 0$, is always a focus or node, and the others are always saddles (see Figure 1).

By a symplectic transformation

$$\begin{cases} x = \sqrt{2\rho}\cos\varphi\,, \\ y = \sqrt{2\rho}\sin\varphi\,, \end{cases}$$

(3) is transformed to the following:

$$(4) \qquad \begin{cases} \dot{\rho} = \mu_1 2\rho + a(\mu)(2\rho)^2 + (2\rho)^{3/2}\cos 3\varphi + (2\rho)^{5/2}F_1(\rho\,, 3\varphi\,, \mu)\,, \\ \dot{\varphi} = \mu_2 + b(\mu)2\rho - (2\rho)^{1/2}\sin 3\varphi + (2\rho)^{3/2}F_2(\rho\,, 3\varphi\,, \mu)\,, \end{cases}$$

where $F_j|_{\mu=0} = 0$, F_j is 2π-periodic with respect to φ, $j = 1\,,2$, $a(\mu) = \operatorname{Re} A(\mu)\,, b(\mu) = \operatorname{Im} A(\mu)\,, |\rho| < \delta_1\,, |\mu| < \delta_1$ and $\delta_1 > 0$ is sufficiently small. We suppose $a(\mu) < 0$ below. The case of $a(\mu) > 0$ is similar.

Let

(5)

$$\mu_1 = -\frac{1}{a(\mu)}\delta^2\beta\,, \mu_2 = -\frac{1}{a(\mu)}\delta\,, \rho \to \frac{\delta^2}{a^2(\mu)}\rho\,, b(\mu) \to -a(\mu)b(\mu)\,, t \to -\frac{a(\mu)t}{\delta}\,,$$

where δ and β are parameters, δ is small and $\beta \in (-\infty\,,\infty)$. Hence, (4) becomes

$$(6) \qquad \begin{cases} \dot{\rho} = \delta\beta(2\rho) - \delta(2\rho)^2 + (2\rho)^{3/2}\cos 3\varphi + \delta^2(2\rho)^{5/2}\overline{F}_1\,, \\ \dot{\varphi} = 1 + b\delta(2\rho) - (2\rho)^{1/2}\sin 3\varphi + \delta^2(2\rho)^{3/2}\overline{F}_2\,. \end{cases}$$

Suppose that $(\rho_o(\delta\,, \beta)\,, \varphi_o(\delta\,, \beta))$ is an equilibrium of (6) which is different from the origin. Let

$$\begin{cases} r = \frac{\rho}{2\rho_o}\,, \\ \theta = \frac{\pi}{6} + \varphi - \varphi_o = \varphi - \psi(\beta\,, \delta). \end{cases}$$

Then (6) takes the form

$$(7) \qquad \begin{cases} \dot{r} = \delta\beta(2r) - \delta(2\rho_o)(2r)^2 + (2\rho_o)^{1/2}(2r)^{3/2}\cos 3(\theta + \psi) + \delta^2(2r)^{5/2}\widetilde{F}_1 \\ \dot{\theta} = 1 + \delta(2\rho_o)b(2r) - (2\rho_o)^{1/2}(2r)^{1/2}\sin 3(\theta + \psi) + \delta^2(2r)^{3/2}\widetilde{F}_2\,. \end{cases}$$

Let $\widetilde{H}(\delta\,, r\,, \theta)$ denote the right-hand side of the equation of $\dot{\theta}$ in (7). We note that the coordinates of the equilibria of (7) are independent of δ and β and these equilibria are: $r = 0$ and $(r_k\,, \theta_k)$, where $r_k = \frac{1}{2}$, $\theta_k = \pi/6 + 2k\pi/3$, $k = 0\,, 1\,, 2$.

For $\delta = 0$, we have $2\rho_o = 1$ and (7) is a Hamiltonian system

$$(8) \qquad \begin{cases} \dot{r} = (2r)^{3/2}\cos 3\theta\,, \\ \dot{\theta} = 1 - (2r)^{1/2}\sin 3\theta\,, \end{cases}$$

with the first integral

$$H(r,\theta) = r - \tfrac{1}{3}(2r)^{3/2}\sin 3\theta. \tag{9}$$

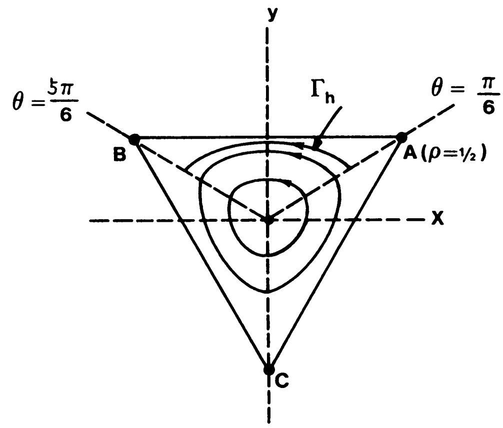

FIGURE 2.

The level curves of $H = h$ are shown in Figure 2, where $0 \le h \le \frac{1}{6}$. $H = 0$ corresponds to the equilibrium $r = 0$, and $H = \frac{1}{6}$ corresponds to the three heteroclinic orbits.

Obviously, any closed orbit of (7) must surround the origin and cross the line segment

$$L = \{(r,\theta)|\theta = \pi/6, 0 \le r \le \tfrac{1}{2}\}.$$

Let

$$H^\delta(r,\theta) = \int_0^r \widetilde{H}(\delta, r, \theta)\, dr.$$

Then (7) can be rewritten in the form:

$$\begin{cases} \dot{r} = -\frac{\partial H^\delta}{\partial \theta} + 2\delta r[\beta - (2\rho_0)(2r) + \delta(2r)^{3/2}\overline{F}], \\ \dot{\theta} = \frac{\partial H^\delta}{\partial r}. \end{cases} \tag{10}$$

We note that $H^\delta(r, \pi/6)$ is monotone in r if $0 \le r \le \frac{1}{2}$. Thus, we parameterize L by $H(r, \pi/6) = h$. Let Γ be a trajectory of (7) starting from a point on L and later intersecting the half line $\theta = 5\pi/6$. We denote by $\Gamma(\delta, h, \beta)$ the part of Γ which lies between the half lines $\theta = \pi/6$ and $\theta = 5\pi/6$. Since (7) is invariant under a rotation through an angle $2\pi/3$, Γ is a closed orbit of (7) if and only if

$$\int_{\Gamma(\delta,h,\beta)} \left(\frac{dH^\delta}{dt}\right) dt = 0. \tag{11}$$

It is easy to obtain from (10) that for $\delta \neq 0$ (11) is equivalent to

$$\Phi(\delta, h, \beta) \equiv \int_{\Gamma(\delta, h, \beta)} r[\beta - (2\rho_0)(2r) + \delta(2r)^{3/2}\overline{F}]\, d\theta = 0. \tag{12}$$

Let Γ_h be the part of level curve $H = h$, $0 \leq h \leq 1/6$, $\pi/6 \leq \theta \leq 5\pi/6$. Define

$$I_k(h) = \int_{\Gamma_h} r^k\, d\theta, \qquad k = 1, 2, 3.$$

In terms of $I_k(h)$ $(k = 1, 2)$, we have

$$\Phi(0, h, \beta) = \int_{\Gamma_h} r(\beta - 2r)\, d\theta = \beta I_1(h) - 2I_2(h). \tag{13}$$

Obviously, $I_1(h) > 0$ for $0 < h < \frac{1}{6}$ and

$$\lim_{h \to 0} \frac{I_2(h)}{I_1(h)} = 0.$$

Let

$$p(h) = \frac{I_2(h)}{I_1(h)}, \qquad 0 \leq h \leq \frac{1}{6}.$$

By using similar arguments as in [2, 4, 5, 8, or 9] one obtains that the uniqueness of limit cycles of (7) is equivalent to the monotonicity of $p(h)$, $0 \leq h \leq \frac{1}{6}$. We will prove the monotonicity of $p(h)$ in Proposition 4 below. The following lemmas are needed.

Lemma 1. *$p(h)$ satisfies the following equation*

$$9h(6h - 1)p'(h) = -12p^2 + (28h - \varphi(h) + 9)p + 48h^2 - 18h + 6h\varphi(h), \tag{14}$$

where $0 < h < \frac{1}{6}$ *and*

$$\varphi(h) = 6h^2(6h - 1)\frac{I_1''(h)}{I_1(h)}.$$

Proof. Suppose that the function $r = r(\theta, h)$ is defined by $H(r, \theta) = h$ for $\pi/6 \leq \theta \leq 5\pi/6$ and $0 \leq h \leq \frac{1}{6}$. From (9) and $H(r, \theta) = h$, we have that

$$\frac{\partial r}{\partial h} = \frac{1}{1 - \sqrt{2r}\sin 3\theta} = \frac{2r}{3h - r} > 0, \qquad 0 < h < \tfrac{1}{6}. \tag{15}$$

The above expression is positive because $0 < r < \frac{1}{2}$. Hence,

$$I_k'(h) = 2k \int_{\Gamma_h} \frac{r^k}{3h - r}\, d\theta.$$

Obviously,

$$I_k(h) = \int_{\Gamma_h} \frac{r^k(3h - r)}{3h - r}\, d\theta = \frac{3h}{2k} I_k' - \frac{1}{2(k+1)} I_{k+1}'.$$

In particular

$$\begin{cases} I_1 = \frac{3}{2}hI_1' - \frac{1}{4}I_2', \\ I_2 = \frac{3}{4}hI_2' - \frac{1}{6}I_3'. \end{cases} \tag{16}$$

From (9), we have

$$I_3(h) = \int_{\Gamma_h} r^3 d\theta = \frac{9}{8}\int_{\Gamma_h} \frac{(r-h)^2}{\sin^2 3\theta}\, d\theta = -\frac{3}{8}\int_{\pi/6}^{5\pi/6} (r-h)^2 d(\cot\theta).$$

Integrating by parts and using (15), (9), and (8), we have

$$\begin{aligned} I_3 &= \frac{3}{4}\int_{\Gamma_h} \frac{r-h}{\sin 3\theta}\,\frac{(2r)^{3/2}\cos^2 3\theta}{1-\sqrt{2r}\sin 3\theta}\, d\theta \\ &= \frac{1}{2}\int_{\Gamma_h} \frac{r[(2r)^3 - 9(r-h)^2]}{3h-r}\, d\theta \\ &= -4I_3 + (\tfrac{9}{2} - 12h)I_2 + (\tfrac{9}{2}h - 36h^2)I_1 + (54h^3 - 9h^2)I_1'. \end{aligned}$$

Hence

$$I_3 = (\tfrac{9}{10} - \tfrac{12}{5}h)I_2 + (\tfrac{9}{10}h - \tfrac{36}{5}h^2)I_1 + (\tfrac{54}{5}h^3 - \tfrac{9}{5}h^2)I_1'. \tag{17}$$

Substituting (17) into (16), we have

$$\begin{cases} 4I_1 = 6hI_1' - I_2', \\ 48I_2 = (18h - 48h^2)I_1' + (-9 + 44h)I_2' - 24h^2(6h-1)I_1''. \end{cases} \tag{18}$$

Equation (18) is equivalent to

$$\begin{cases} 9h(6h-1)I_1' = (-9+44h)I_1 + 12I_2 + 6h^2(6h-1)I_1'', \\ 9h(6h-1)I_2' = (-18h + 48h^2)I_1 + 72hI_2 + 36h^3(6h-1)I_1'', \end{cases} \tag{19}$$

for $0 < h < \frac{1}{6}$. By (19) and the following

$$p'(h) = \frac{I_2'I_1 - I_1'I_2}{I_1^2}$$

we obtain (14). This proves the lemma. □

Lemma 2. $\lim_{h\to 0} p'(h) = 1$.

Proof. As $h \to 0$, $I_1 = O(h)$, $I_2 = O(h^2)$ (see (15)). Hence, $p(h) = O(h)$ and

$$I_1''(h) = 6\int_{\Gamma_h} \frac{r(r-h)}{(3h-r)^3}\, d\theta = O(|h|^{-1/2})$$

$$I_1'''(h) = 12\int_{\Gamma_h} \frac{r(r-h)(5r-6h)}{(3h-r)^5}\, d\theta = O(|h|^{-3/2}).$$

Thus, $\varphi(h) = O(|h|^{1/2})$, $\varphi'(h) = O(|h|^{-1/2})$. Using the above estimates and L'Hospital's rule, we obtain from (14) $\lim_{h\to 0} p'(h) = 1$. □

Lemma 3. $p(\frac{1}{6}) = \frac{1}{16}$, $p'(\frac{1}{6}) = 2\sqrt{5}\ln((3+\sqrt{5})/2) - 4$.

Proof. $\Gamma_{1/6}$ is a curve given by $\{(r,\theta) | \sqrt{2r}\sin\theta = 1/4,\ \pi/6 \le \theta \le 5\pi/6\}$. A direct calculation shows

$$I_1\left(\frac{1}{6}\right) = \int_{\Gamma_{1/6}} r\, d\theta = \frac{\sqrt{3}}{16}, \qquad I_2\left(\frac{1}{6}\right) = \int_{\Gamma_{1/6}} r^2 d\theta = \frac{\sqrt{3}}{256},$$

$$I_1'\left(\frac{1}{6}\right) = \int_{\Gamma_{1/6}} \frac{2r}{\frac{1}{2}-r}\,d\theta = \frac{2}{\sqrt{15}}\ln\frac{3+\sqrt{5}}{2},$$

and

$$I_2'\left(\frac{1}{6}\right) = \int_{\Gamma_{1/6}} \frac{4r^2}{\frac{1}{2}-r}\,d\theta = \frac{\sqrt{3}}{4}\left[\frac{8\sqrt{5}}{15}\ln\frac{3+\sqrt{5}}{2} - 1\right].$$

Hence

$$p(1/6) = \left.\frac{I_2}{I-1}\right|_{h=1/6} = \frac{1}{16}$$

$$p'(1/6) = \left.\frac{I_2'I_1 - I_1'I_2}{I_1^2}\right|_{h=1/6} = 2\sqrt{5}\ln\frac{3+\sqrt{5}}{2} - 4. \quad \square$$

Proposition 4. $p'(h) > 0$ *for* $0 < h < \frac{1}{6}$.

Proof. We will prove that if there exists $h_0 \in (0, 1/6)$ such that $p'(h_0) = 0$, then $p''(h_0) > 0$. This is a contradiction since $p(0) = 0$, $p'(0) = 1$ (Lemma 1).

Let

$$Q(h) = I_2'(h)/I_1'(h), \qquad 0 < h < \tfrac{1}{6}.$$

Then $p(h_0) = Q(h_0)$ and

$$p''(h_0) = (I_1'(h_0)/I_1(h_0))Q'(h_0)$$

provided $p'(h_0) = 0$ (see Carr, Chow, and Hale [2]).

From the first equation of (18), we have

$$6hI_1'' = I_2'' - 2I_1'.$$

Hence

$$\frac{I_1''}{I_1'} = \frac{1}{6h}\left(\frac{I_2''}{I_1'} - 2\right) = \frac{1}{6h}\left(\frac{I_2''}{I_2'}Q - 2\right).$$

This implies

$$Q'(h) \equiv Q\left(\frac{I_2''}{I_2'} - \frac{I_1''}{I_1'}\right) = \frac{Q}{6h}\left[(6h-Q)\frac{I_2''}{I_2'} + 2\right].$$

The first equation of (18) implies $6h - Q = 4I_1/I_1'$.

If $p'(h_0) = 0$ $(0 < h_0 < \frac{1}{6})$, then $Q(h_0) = p(h_0) > 0$ and

$$\frac{I_1(h_0)}{I_1'(h_0)} = \frac{I_2(h_0)}{I_2'(h_0)}.$$

Thus

$$Q'(h_0) = \frac{Q(h)}{3h}\left(\frac{2I_2I_2''}{I_2'^2} + 1\right)\Bigg|_{h=h_0} > 0$$

because

$$I_2(h_0) = \int_{\Gamma_{h_0}} r^2\,d\theta > 0,$$

$$I_2'(h_0) = \int_{\Gamma_{h_0}} \frac{4r^2}{3h_0 - r}\, d\theta > 0\,,$$

$$I_2''(h_0) = \int_{\Gamma_{h_0}} \frac{4r^2(3h_0 + r)}{(3h_0 - r)^3}\, d\theta > 0.$$

The fact that along Γ_{h_0}, $3h_0 - r > 0$ can be found from (15). This proves Proposition 4. □

Remark. We note that all the above discussions are independent of the higher order terms $O(|z|^4)$.

References

1. V. I. Arnold, *Geometrical methods in the theory of ordinary differential equations*, Springer-Verlag, New York, 1983.
2. J. Carr, S.-N. Chow, and J. K. Hale, *Abelian integrals and bifurcation theory*, J. Differential Equations **59** (1985), 413–436.
3. S.-N. Chow, C. Li, and D. Wang, *Normal forms and bifurcation of vector fields* (to appear).
4. R. H. Cushman and J. A. Sanders, *A codimension two bifurcation with a third order Picard-Fuchs equation*, J. Differential Equations **59** (1985), 243–256.
5. B. Drachman, S. van Gils, and Zhang Zhifen, *Abelian integrals for quadratic vector fields*, J. Reine Angew. Math. **382** (1987), 165–180.
6. E. I Horozov, *Versal deformation of equivariant vector fields in the case of symmetry of order* 2 *and* 3, Trans. of Petrovski Seminar, **5** (1979), 163–192, (In Russian).
7. C. Rousseau, *Codimension* 1 *and* 2 *bifurcations of fixed points of diffeomorphisms and of periodic solutions of vector fields*, 1987 (preprint).
8. R. H. Cushman and J. A. Sanders, *Abelian integrals and global Hopf bifurcations*, Dynamical Systems and Bifurcations (B. L. J. Braaksma, H. W. Broer, and F. Takens, editors) Lecture Notes in Mathematics 1125, Springer-Verlag, 1985, pp. 87–98.
9. Yu. S., Il′yashenko, *Zeros of special Abelian integrals in a real domain*, Functional Anal. Appl. **11** (1977), 309–311.

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