

THERE'S NO FORCING A LEAST UPPER BOUND

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ABSTRACT. While it is possible to have an increasing ω -sequence of c -degrees with a least upper bound in a particular model of ZF, we show that it is not possible to force a sequence with a least upper bound absolute among all generic extensions of the original model

In [H], Hájek forces over L a sequence of reals $\langle a_n \mid n \in \omega \rangle$ which, when coded in some trivial manner into a real α , is a good candidate for the least upper bound of the c -degrees of the a_n 's. "Good candidate" means that any upper bound which does not collapse \aleph_1 also constructs the sequence. He left open the question of whether it is actually the least upper bound. We show here that it is not, which means that there is no least upper bound.

Hájek's partial order is the ω -step iteration (with finite support) of the Jensen-Johnsbraten forcing [JJ]. The absoluteness comes about because any pair of JJ-generics (over the same ground model satisfying $V = L$) collapses \aleph_1 . Also, a JJ-generic G over L is definable in $L[G]$. So if $\aleph_1^V = \aleph_1^L$ then the sequence of generics is definable as the ω -sequence $\langle a_0, a_1, \dots \rangle$ such that a_{n+1} is the unique set satisfying the appropriate definition over $L[a_n]$.

The reason that Hájek's sequence is not necessarily definable if \aleph_1^L is collapsed is that for each n there may be many such generics with no way of distinguishing a_{n+1} from the rest. More particularly, there may be a perfect tree the n th level of which is a set of mutually generic reals for the n th partial order. Even if Hájek's generic is a path through this tree, there is no guarantee that it is constructible from the tree.

The theorem below is a general statement which implies as a particular instance that Hájek's sequence $\langle a_0, a_1, \dots \rangle$ has no l.u.b. We chose this presentation in part because it focuses attention on the relevant points, which do not include special features of a partial order. This result is actually just another application of the main idea of [L] of hiding a generic as a path through a tree. There, though, it was used in somewhat the opposite direction, showing that a particularly recalcitrant set of hyperdegrees does have a minimal upper bound.

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Definition 1. For $\mathcal{M} \models \text{ZF}$ standard, $A \leq_{\mathcal{M}} B$ if $A \in \mathcal{M}[TC(B)]$. The equivalence classes of $\leq_{\mathcal{M}}$ are the \mathcal{M} -degrees. By convention, we restrict our attention to standard sets A such that $\mathcal{M}[TC(A)] \models \text{ZF}$.

Theorem 2. Let \mathcal{M} be a countable standard model of ZF, $\mathcal{P}_{\omega} = \lim_{n \in \omega} \mathcal{P}_n \in \mathcal{M}$ a limit of complete Boolean algebras, and $\langle G_0, G_1, \dots \rangle$ a \mathcal{P}_{ω} -generic over \mathcal{M} , with G_n \mathcal{P}_n -generic and $G_{n+1} \not\leq_{\mathcal{M}} G_n$. Then $\{\mathcal{M} - \deg(G_n) \mid n \in \omega\}$ has no least upper bound in the \mathcal{M} -degrees.

Remarks. In the statement of the theorem, the kind of limit taken is left ambiguous, since it does not matter. Note that $\mathcal{P}_{\omega} \neq \mathcal{P}_0 * \mathcal{P}_1 * \dots$. Rather $\mathcal{P}_{\omega} = \mathcal{Q}_0 * \mathcal{Q}_1 * \dots$ and $\mathcal{P}_n = \mathcal{Q}_0 * \dots * \mathcal{Q}_n$ for an appropriate choice of \mathcal{Q}_i 's. We freely identify \mathcal{P}_n with its canonical image in \mathcal{P}_{ω} or \mathcal{P}_m ($m > n$). We assume that $1 \Vdash \text{"}\mathcal{Q}_n \text{ is a complete atomless Boolean algebra"}$, by working beneath a condition forcing such. Our final forcing partial order \mathcal{P} is an ω iteration, but is neither a direct nor an inverse limit. For more details on forcing and limits in the language of Boolean algebra, see [J].

The idea of the proof is to take the sequence $\langle G_0, G_1, \dots \rangle$ and hide it in a bigger model. This is done by including many \mathcal{Q}_{n+1} -generics over $\mathcal{M}[G_n]$, to keep G_{n+1} from being a canonical choice, even with parameter G_n . Thus we want a tree of order type $\omega^{<\omega}$, and at node σ a generic over $\mathcal{P}_{|\sigma|}$ which (except for $\sigma = \emptyset$) extends the generic for $\sigma \upharpoonright |\sigma| - 1$, such that $\langle G_0, G_1, \dots \rangle$ is a "generic" path through the tree. Thus the tree itself is an upper bound for $\{\mathcal{M} - \deg(G_n) \mid n \in \omega\}$, since each G_n is a node on the tree. Furthermore, the tree does not construct a previously given upper bound G_{ω} by the "genericity" of the path $\langle G_0, G_1, \dots \rangle$.

Such a tree can be constructed generically, using the partial order \mathcal{P} with conditions p of partial functions having domain $\omega^{<\omega}$ and $p(\sigma) \in \mathcal{P}_{|\sigma|}$. There are conflicting demands upon \mathcal{P} , however. \mathcal{P} must include all conditions in \mathcal{P}_{ω} , which might necessitate conditions with infinite support. It must also have some sort of finite support, in order to allow for the appropriate diagonalization. (Such diagonalization is the "genericity" of $\langle G_0, G_1, \dots \rangle$ as a path through the \mathcal{P} -generic, as referred to above.) Thus a condition p will have as support finitely many paths through $\omega^{<\omega}$.

Definition 3. Working in \mathcal{M} , let \mathcal{P} be $\{p: \omega^{<\omega} \rightarrow \bigcup_{n \in \omega} \mathcal{P}_n \mid p \text{ is partial and for some } f_1, \dots, f_n \in \omega^{\omega}, n \text{ finite}$

- (a) $\text{dom } p = \{f_i \upharpoonright j \mid i \leq n, j < \omega\}$
- (b) $p(\sigma) \in \mathcal{P}_{|\sigma|}$
- (c) $\pi_j^{j+1}(p(f_i \upharpoonright j + 1)) = p(f_i \upharpoonright j)$ and
- (d) in \mathcal{P}_{ω} , $\lim_{j \rightarrow \omega} p(f_i \upharpoonright j) \neq \mathcal{P}_{\omega} 0$. (π_l^k is the canonical projection from \mathcal{P}_k to \mathcal{P}_l .)

Let $q \leq p$ if $\text{dom}(q) \supseteq \text{dom}(p)$, and $q(\sigma) \leq_{\mathcal{P}_{|\sigma|}} p(\sigma)$ for $\sigma \in \text{dom } p$.

Let G_{ω} be an upper bound of the G_n 's in the \mathcal{M} -degrees. We want a \mathcal{P} -generic G such that:

- (a) $\forall n \exists \sigma G_n = \{p(\sigma) \mid p \in G\}$ and
 (b) $G_\omega \notin \mathcal{M}[TC(G)]$.

Notice that such a G is an upper bound for $\{\mathcal{M} - \deg(G_n) \mid n \in \omega\}$ by (a), which does not construct an alleged l.u.b. G_ω , thus proving the theorem.

The following definitions and lemma are useful, and quite natural.

Definition 4. For $\sigma \in \omega^{<\omega}$, let $G(\sigma) = \{p(\sigma) \mid p \in G\}$. (Thus condition (a) above is " $\forall n \exists \sigma G_n = G(\sigma)$ ".)

We want to take the Boolean meet of conditions, even if they come from different Boolean algebras. Hence,

Definition 5. For $\sigma, \tau \in \omega^{<\omega}$, let $m(\sigma, \tau) = \sup\{m \mid \sigma \upharpoonright m = \tau \upharpoonright m\}$, the length of agreement of σ and τ .

For $p \in \mathcal{P}$, $r \in \mathcal{P}_n$, $\sigma \in \omega^n$, let $p \wedge_\sigma r \in \mathcal{P}$ have domain $\text{dom } p \cup \{\sigma \upharpoonright m \mid m \leq |\sigma|\}$, and have $(p \wedge_\sigma r)(\tau) = p(\tau) \wedge \pi_{m(\sigma, \tau)}^n(r)$. Notice that $p \wedge_\sigma r$ is undefined if $p(\sigma) \perp r$, by clause (d) in the definition of \mathcal{P} .

Lemma 6. For G \mathcal{P} -generic, $G(\sigma)$ is $\mathcal{P}_{|\sigma|}$ -generic.

Proof. Suppose $D \subseteq \mathcal{P}_{|\sigma|}$ in \mathcal{M} is dense. Given $p \in \mathcal{P}$, let $r \leq p(\sigma)$, $r \in D$. $p \wedge_\sigma r$ as defined above is a condition, because if

$$\lim_{j \rightarrow \omega} ((p \wedge_\sigma r)(f_i \upharpoonright j)) = \pi_{m(f_i, \sigma)}^{|\sigma|}(r) \wedge \lim_{j \rightarrow \omega} p(f_i \upharpoonright j) = 0,$$

then $\lim_{j \rightarrow \omega} p(f_i \upharpoonright j) \leq \neg[\pi_{m(f_i, \sigma)}^{|\sigma|}(r)]$. But then

$$p(f_i \upharpoonright m(f_i, \sigma)) \leq \neg[\pi_{m(f_i, \sigma)}^{|\sigma|}(r)],$$

which contradicts $r \leq p(\sigma)$. So $p \wedge_\sigma r \leq p$ and $(p \wedge_\sigma r)(\sigma) \in D$, which means that generically $G(\sigma)$ will hit each dense $D \subseteq \mathcal{P}_{|\sigma|}$ from \mathcal{M} . \square

Now we construct G \mathcal{P} -generic over \mathcal{M} , satisfying (a) and (b). The construction takes ω -many stages. The first step at stage n is to hit the n th dense set, and in such a way that G_n remains compatible with (the current state of) $G(\sigma_n)$ (some σ_n suitably chosen). The second step is the diagonalization, to ensure that the n th term, when finally evaluated in $M[G]$, does not equal G_ω (again respecting G_n).

Let $\langle D_n \mid n \in \omega \rangle$, $\langle tm_n \mid n \in \omega \rangle$ list, respectively, the dense subsets of \mathcal{P} in \mathcal{M} and the terms for members of $\mathcal{M}[G]$ in \mathcal{M} .

Start with $\sigma_0 = \emptyset$ and $p_0 = \langle \phi, 1_{\mathcal{P}_0} \rangle$. Assume inductively that at stage n we have $p_n \in \mathcal{P}$ and $\sigma_n \in \text{dom } p_n$ of length n so that $p_n(\sigma_n) \in G_n$.

The following lemma will be used several times. It justifies extending a condition $p \in \mathcal{P}$ to satisfy a density requirement while retaining the compatibility of $G(\sigma)$ and $G_{|\sigma|}$ for some given σ .

Lemma 7. Let $p \in \mathcal{P}$, $\sigma \in \text{dom } p$, and $D \subseteq \mathcal{P}$ dense and in \mathcal{M} . Then $\{p^\sigma \in \mathcal{P} \mid \exists q \in D q \leq p \text{ and } q(\sigma) = p^\sigma\}$ is dense beneath $p(\sigma)$ in $\mathcal{P}_{|\sigma|}$.

Proof of Lemma. Let $r \leq_{\mathcal{P}_{|\sigma|}} p(\sigma)$. Extend $p \wedge_\sigma r$ to $q \in D$. $q(\sigma) \leq r$ is as desired. \square

To complete the first step of stage n , apply Lemma 7 with $p = p_n$, $\sigma = \sigma_n$, and $D = D_n$. By genericity of G_n , let $p'_n \leq p_n$, $p'_n \in D_n$, $p'_n(\sigma_n) \in G_n$.

Now apply Lemma 7 with $p = p'_n$, $\sigma = \sigma_n$, and $D = \{p \mid p \Vdash "tm_n \in \mathcal{M}[G(\sigma_n)]"\}$, to get $p''_n \leq p'_n$ deciding " $tm_n \in \mathcal{M}[G(\sigma_n)]$ ", $p''_n(\sigma_n) \in G_n$. If $p''_n \Vdash "tm_n \in \mathcal{M}[G(\sigma_n)]"$, then let p'''_n be p''_n . If $p''_n \Vdash "tm_n \notin \mathcal{M}[G(\sigma_n)]"$, then we want to apply Lemma 7 with $p = p''_n$, $\sigma = \sigma_n$, and $D = D_{tm_n} =_{\text{def}} \{q \mid q \perp p''_n \text{ or there are } q_1, q_2 \leq q \text{ } q_1(\sigma_n) = q_2(\sigma_n) = q(\sigma_n) \text{ and } q_1, q_2 \text{ are } tm_n\text{-incompatible (see below)}\}$.

Lemma 8. D_{tm_n} is dense.

Remark. " tm_n -incompatible" means "forces contradictory statements about tm_n ": $q_1 \Vdash \varphi$ and $q_2 \Vdash \neg\varphi$, for some φ all of whose terms are canonical names for members of \mathcal{M} or tm_n itself.

Proof. Let $\tilde{q} \leq p''_n$. Let G be \mathcal{P} -generic. $G(\sigma_n)$ is \mathcal{P}_n -generic by Lemma 6, so $\mathcal{P} \simeq \mathcal{P}_n * \mathcal{P}/G(\sigma_n)$. Furthermore, by hypothesis on p''_n , $tm_n^{\mathcal{M}[G]} \notin \mathcal{M}[G(\sigma_n)]$. So there are tm_n -incompatible q'_1, q'_2 in $\mathcal{P}/G(\sigma_n)$, since otherwise $tm_n^{\mathcal{M}[G]}$ would be definable in $\mathcal{M}[G(\sigma_n)]$ as $\{x \mid \text{for some } r \in \mathcal{P}/G(\sigma_n), r \Vdash "\dot{x} \in tm_n"\}$. Let $q' \in G(\sigma_n)$ force such about q'_1 and q'_2 . Then $q = \tilde{q} \wedge_{\sigma_n} q'$ is as desired, with witnesses $q_1 = \langle q(\sigma_n), q \wedge q'_1 \rangle$ and $q_2 = \langle q(\sigma_n), q \wedge q'_2 \rangle$. \square

By genericity of G_n , let $p_n^{''1/2} \leq p'_n$ be such that $p_n^{''1/2}(\sigma_n) \in G_n$, and $p_n^{''1/2} \in D_{tm_n}$ with witnesses q_1 and q_2 . Let p'''_n be a q_i which forces something about tm_n false for G_ω . That is, for φ the witness to the tm_n -incompatibility of q_1 and q_2 , and $\psi = \varphi$ or $\neg\varphi$, p'''_n is the q_i such that $q_i \Vdash \psi$ and $M[G_\omega] \models \neg\psi$ (with tm_n interpreted as G_ω in the latter context).

Choose an m so that $\sigma_n^\wedge m \notin \text{dom } p'''_n$. Let $\sigma_{n+1} = \sigma_n^\wedge m$ and $p_{n+1} = p'''_n \cup \{(\sigma_{n+1}^\wedge 0^t, p'''_n(\sigma_n)) \mid t \in \omega\}$. This completes the induction.

Let $G = \{p \mid p \geq p_n \text{ for some } n\}$. Notice that for $m > n$, σ_m extends σ_n , so $p_m(\sigma_m) \leq G_m$ implies $p_m(\sigma_n) \in G_n$. Consequently, by the first step in the construction, $G(\sigma_n) \subseteq G_n$. Each is \mathcal{P}_n -generic. Therefore $G(\sigma_n) = G_n$. So G is an upper bound of the G_n 's. By the diagonalization of the second step, $G_\omega \notin \mathcal{M}[G]$. (Note that in the first case of the second step, $tm_n^{\mathcal{M}[G]} \in \mathcal{M}[G(\sigma_n)] = \mathcal{M}[G_n]$, and $G_\omega \notin \mathcal{M}[G(\sigma_n)]$ for any n since $G_{n+1} \notin \mathcal{M}[G_n]$ by hypothesis.) Therefore G_ω is not the least upper bound of the G_n 's. \square

Question. $0^{\#\omega} = \text{l.u.b. } 0^{\#n}$, so least upper bounds are possible. Is there a countable standard model of ZF with an absolute least upper bound (that is, a degree which is an l.u.b. in all models with the same ordinals) that does not collapse cardinals?

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