

HAUSDORFF MEASURES AND SETS OF UNIQUENESS FOR TRIGONOMETRIC SERIES

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ABSTRACT. We characterize the closed sets E in the unit circle T which have the property that, for some nondecreasing $h: (0, \infty) \rightarrow (0, \infty)$ with $h(0+) = 0$, all the Hausdorff h -measure 0 closed sets $F \subseteq E$ are sets of uniqueness (for trigonometric series). In conjunction with Körner's result on the existence of Helson sets of multiplicity, this implies the existence of closed sets of multiplicity (M -sets) within which Hausdorff h -measure 0 implies uniqueness, for some h . This is contrasted with the case of closed sets of strict multiplicity (M_0 -sets), where results of Ivashev-Musatov and Kaufman establish the opposite.

A *Hausdorff determining function* is any function $h: (0, \infty) \rightarrow (0, \infty)$ which is nondecreasing and satisfies $h(0+) = 0$. A subset E of T (the *unit circle*, viewed here as $\mathbf{R}/2\pi\mathbf{Z}$) has *Hausdorff h -measure 0* if for every $\varepsilon > 0$ there is a sequence $\{I_n\}$ of open intervals (arcs) in T with $E \subseteq \bigcup_n I_n$ and $\sum h(|I_n|) < \varepsilon$. (Here $|I_n| = (\text{arc length of } I_n)/2\pi$.) It is a well-known theorem of Ivashev-Musatov [2] that metric thinness in the form of Hausdorff h -measure 0 cannot imply uniqueness for trigonometric series. More precisely, for any h as above there is a closed set E of Hausdorff h -measure 0 which is of restricted multiplicity (we will give a review of terminology below). Kaufman [3; see also 4, VIII.3.3] extended this result by proving that, for any h as above, any closed set of strict multiplicity has a closed subset of Hausdorff h -measure 0 which is still of restricted multiplicity. Later Kaufman asked (in a private conversation) whether the same holds for ordinary multiplicity. The main result of this paper gives a negative answer to Kaufman's question, and provides a characterization of those closed sets which, for some h , contain no closed subsets of Hausdorff h -measure 0 which are of multiplicity.

We now give a brief review of the notation, terminology, and results we need; this material all appears in Kechris-Louveau [4]. Let $K(T)$ be the collection of closed subsets of T . Denote by U the collection of sets $E \in K(T)$ which are of *uniqueness* (i.e., every trigonometric series converging to 0 on $T \setminus E$ is identically 0), and let $M = K(T) \setminus U$ (the collection of closed sets of *multiplicity*). This is

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the classical definition. It can be reformulated in the theory of distributions as follows: A closed set $E \in K(\mathbb{T})$ is in M if it supports no non-0 pseudomeasure whose n th Fourier coefficient tends to 0 as $n \rightarrow \pm\infty$. Let $U_0 \subseteq K(\mathbb{T})$ be the collection of sets of extended uniqueness (i.e., those which support no probability measure whose n th Fourier coefficient tends to 0 as $n \rightarrow \pm\infty$) and let $M_0 = K(\mathbb{T}) \setminus U_0$ be the collection of closed sets of *restricted multiplicity*.

Piatetski-Shapiro gave a characterization of sets of uniqueness in terms of Banach spaces. Let A be the space of continuous complex functions on \mathbb{T} with absolutely convergent Fourier series (which is isomorphic to $l_1(\mathbb{Z}) \cong c_0(\mathbb{Z})^*$). For $E \in K(\mathbb{T})$, let $I(E)$ ($J(E)$) be the set of members of A which vanish on E (on a neighborhood of E). In general, $I(E) \neq J(E)$. Then E is in U iff $J(E)$ is weak*-dense in A (originally in [7]; see also [4, V.4.1]). Let U_1 be the collection of $E \in K(\mathbb{T})$ such that $I(E)$ is weak*-dense in A ; then $U \subseteq U_1 \subseteq U_0$, and Körner showed that $U_1 \neq U$ (originally in [6]; see also [4, VII.3]).

The sets U and U_0 are closed under countable unions within $K(\mathbb{T})$ (Bari's Theorem; see [4, I.5.1 and II.5]), but U_1 is not [4, VI.3.6]; let U_1^* be the set of $E \in K(\mathbb{T})$ which are countable unions of sets in U_1 . Piatetski-Shapiro showed that every set in U_1 (and hence every set in U_1^*) is a countable union of sets in U_1' , where $E \in K(\mathbb{T})$ is in U_1' iff $I(E)$ is sequentially weak*-dense in A (originally in [7; see also 4, VI.2.3]). One has now the following analogue of Kaufman's result: if h is a Hausdorff determining function and $E \in K(\mathbb{T}) \setminus U_1^*$, then there is $F \subseteq E$ in M with Hausdorff h -measure 0 [4, VIII.4.8]. (This is indeed a corollary of a much more general result: If $E \in K(\mathbb{T})$ is such that for every open $V \subseteq \mathbb{T}$ with $E \cap V \neq \emptyset$ we have $\overline{E \cap V} \notin U_1$, then every dense G_δ subset of E contains a closed set F in M .) Kaufman asked whether one can replace $E \in K(\mathbb{T}) \setminus U_1^*$ by $E \in M$; we will answer this question negatively by showing that the above property characterizes U_1^* .

Theorem. *A set $E \in K(\mathbb{T})$ is in U_1^* iff there is a Hausdorff determining function h such that every closed subset of E of Hausdorff h -measure 0 is in U .*

One direction of this has already been noted; for the other, we must show that for every $E \in U_1^*$ such an h exists. We will use the idea of a construction from Debs-Saint Raymond [1].

Let S be a countable collection of infinite sequences $\{a_n\}_{n=1}^\infty$ of positive reals. We say that $E \subseteq \mathbb{T}$ is *S-small* if, for each $\{a_n\} \in S$, there is a sequence $\{I_n\}$ of open intervals such that $E \subseteq \bigcup_n I_n$ and $|I_n| \leq a_n$. Note that, for every Hausdorff determining function h , there is a collection S such that any *S-small* $E \subseteq \mathbb{T}$ has Hausdorff h -measure 0; just let the m th sequence in S be $\{a_n^{(m)}\}$ where $0 < h(a_n^{(m)}) < 2^{-n-m}$. Conversely, we have:

Lemma. *Given any S as above, there is an h such that any $E \subseteq \mathbb{T}$ which has Hausdorff h -measure 0 is *S-small*.*

Proof. Let the m th sequence in S be $\{a_n^{(m)}\}$, and choose h so that, for all n and m , $h(a_n^{(m)}) > 1/mn$. Suppose E has Hausdorff h -measure 0. Let $\{I_k\}$ be open intervals with $E \subseteq \bigcup_k I_k$ and $\sum_k h(|I_k|) < 1/m$. We may assume $|I_1| \geq |I_2| \geq \dots$. Then $|I_n| \leq a_n$, because $|I_n| > a_n$ would imply $h(|I_n|) \geq h(a_n) > 1/mn$ and $h(|I_n|) \geq h(|I_2|) \geq \dots \geq h(|I_n|) > 1/mn$, so $\sum_k h(|I_k|) > n \cdot (1/mn) = 1/m$, a contradiction. So E is S -small.

So assuming $E \in U_1^*$, it is enough to find S so that

$$(*) \quad F \subseteq E, F \text{ closed}, F \text{ } S\text{-small} \Rightarrow F \in U.$$

Since U is closed under countable unions, it is enough to show $(*)$ assuming $E \in U_1'$. So suppose $f_m \in I(E)$ and $f_m \xrightarrow{w^*} 1$.

Lemma. Let $f \in A$, $\varepsilon > 0$. There is a sequence $\{a_n\}$ of positive reals such that if $F \subseteq Z(f) = \{x \in \mathbf{T} : f(x) = 0\}$ and $F \subseteq \bigcup_n I_n$, where I_n are open intervals with $|I_n| \leq a_n$, then for some $g \in J(F)$, $\|f - g\|_A < \varepsilon$.

Granting the lemma we can complete the proof as follows: Let $\{a_n^{(m)}\}_{n=1}^\infty$ be the sequence given by the lemma for $f = f_m$, $\varepsilon = 1/m$. If the m th sequence in S is $\{a_n^{(m)}\}$, and $F \subseteq E$ is S -small, then there is $g_m \in J(F)$ with $\|f_m - g_m\|_A \leq 1/m$. So $g_m \xrightarrow{w^*} 1$ and hence $J(F)$ is (actually sequentially) weak $*$ -dense in A , so $F \in U$.

Proof of the lemma. Let $\tau_{x,\delta}$ be the usual trapezoidal function, defined to be 1 on $[x - \delta, x + \delta]$, 0 outside $(x - 2\delta, x + 2\delta)$, and linear on $[x - 2\delta, x - \delta]$ and $[x + \delta, x + 2\delta]$. For each $g \in A$, let

$$N_g(\varepsilon) = \min \left\{ N \geq 1 : \sum_{|n| \geq N} |\hat{g}(n)| < \varepsilon \right\}.$$

It is not hard to show that

$$N_{g_1 g_2}(\varepsilon) \leq N_{g_1} \left(\frac{\varepsilon}{2\|g_2\|_A} \right) + N_{g_2} \left(\frac{\varepsilon}{2\|g_1\|_A} \right)$$

for any $g_1, g_2 \in A$. We will now show that, if $g \in A$, $\delta \in (0, 1)$, and $g(x) = 0$, then

$$\delta < \frac{K\varepsilon^2}{\|g\|_A^2 \cdot N_g(\varepsilon)^2} \Rightarrow \|g\tau_{x,\delta}\|_A < 8\varepsilon$$

for some fixed constant K . What is at issue here is that “points obey spectral synthesis.” For the reader not familiar with these ideas, we provide the details. First recall the result of Bernstein that, for any trigonometric polynomial Q of degree at most N , $\|Q'\|_\infty \leq 2N\|Q\|_\infty$. (One proves this by showing that Q' is the convolution of $-2NQ$ with $K_{N-1}(y) \sin Ny$, where $K_N(y) = \sum_{|j| \leq N} (1 - |j|/(N+1))e^{ijy}$ is the Féjer kernel.) Also, $\|\tau_{x,\delta}\| \leq 3$ [4, p. 58] and,

for any $f \in C(T)$ with $f' \in L^2(T)$, $\|f\|_A \leq |\hat{f}(0)| + C\|f'\|_{L^2}$ for some fixed constant C [4, II.2.1]. Now let

$$Q(y) = \sum_{n \leq |N|} \hat{g}(n)e^{iny} + \sum_{n > |N|} \hat{g}(n)e^{inx};$$

then Q is a trigonometric polynomial of degree $\leq N$, $\|Q\|_\infty \leq \|g\|_A$, $\|g - Q\|_A \leq 2 \sum_{|n| > N} |\hat{g}(n)|$, $\|g\tau_{x,\delta}\|_A \leq \|Q\tau_{x,\delta}\|_A + \|g - Q\|_A \|\tau_{x,\delta}\|_A$, $\|Q\tau_{x,\delta}\|_A \leq \|Q\tau_{x,\delta}\|_1 + C\|(Q\tau_{x,\delta})'\|_2$, $\|Q\tau_{x,\delta}\|_1 \leq (4\delta/2\pi)\|Q\|_\infty$, $|Q| \leq 2\delta\|Q'\|_\infty$ on $[x - 2\delta, x + 2\delta]$ (since $Q(x) = 0$), $(Q\tau_{x,\delta})' = Q\tau'_{x,\delta} + Q'\tau_{x,\delta}$, $\|Q\tau'_{x,\delta}\|_2 \leq K_1\delta^{-1/2} \sup_{y \in [x-2\delta, x+2\delta]} |Q(y)| \leq 2K_1\|Q'\|_\infty\sqrt{\delta}$, and $\|Q'\tau_{x,\delta}\|_2 \leq K_2\|Q'\|_\infty\delta$; putting these together gives the desired result.

It follows that one can inductively define a_1, a_2, \dots , such that, for any $x_1, x_2, \dots \in Z(f)$, if we let $f_0 = f$, $f_1 = f_0 \cdot (1 - \tau_{x_1, a_1})$, $f_2 = f_1 \cdot (1 - \tau_{x_2, a_2}) = f_0 \cdot (1 - \tau_{x_1, a_1}) \cdot (1 - \tau_{x_2, a_2})$, \dots , $f_{n+1} = f_n \cdot (1 - \tau_{x_{n+1}, a_{n+1}})$, \dots , then $\|f_n - f_{n+1}\|_A < \varepsilon/2^{n+1}$.

If now $F \subseteq Z(f)$ and $F \subseteq \bigcup_n I_n$, where I_n is an open interval with $|I_n| \leq a_n$, let k be large enough so that $F \subseteq \bigcup_{n \leq k} I_n$. We can also clearly assume $F \cap I_n \neq \emptyset$ for $n \leq k$. Choose $x_n \in F \cap I_n$. Let

$$J_n = (x_n - a_n, x_n + a_n).$$

Then $x_n \in Z(f)$ and $F \subseteq \bigcup_{n \leq k} J_n$. Let $g = f \cdot (1 - \tau_{x_1, a_1})(1 - \tau_{x_2, a_2}) \cdots (1 - \tau_{x_k, a_k})$. Then $g = 0$ on $\bigcup_{n \leq k} J_n$, so $g \in J(F)$, and $\|f - g\|_A \leq \sum_n \|f_n - f_{n+1}\|_A < \varepsilon$.

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