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THE RANK IN HOMOGENEOUS SPACES OF NONPOSITIVE CURVATURE

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ABSTRACT. Given a solvable and simply connected Lie group G with Lie algebra $\mathfrak{g}$ and a left invariant metric of nonpositive curvature without flat factor, we prove that $\operatorname{rank}(G) \leq \dim \mathfrak{a}$, where $\mathfrak{a}$ is the orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$. In particular, if H is a simply connected homogeneous space of nonpositive curvature satisfying the visibility axiom then H has rank one.

INTRODUCTION

Let G be a solvable and simply connected Lie group with a left invariant metric of nonpositive curvature. If $\mathfrak{g}$ is the Lie algebra of G, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{a}$ where $\mathfrak{a}$, the orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$ with respect to the metric, is an abelian subalgebra of $\mathfrak{g}$.

The *rank of* G ($\operatorname{rank}(G)$) is defined as the minimum of the dimensions of the spaces of parallel Jacobi vector fields along the geodesics through the identity of G. This definition coincides with the usual one in the symmetric case.

In this paper we show that if G does not have de Rham flat factor, then $\operatorname{rank}(G)$ is at most $\dim \mathfrak{a}$ (Theorem 1.3). This bound for $\operatorname{rank}(G)$ is the best possible since for symmetric G, it coincides with $\dim \mathfrak{a}$ (Remark 1.4).

As a consequence, we obtain that if H is a simply connected homogeneous space of nonpositive curvature satisfying the visibility axiom then H has rank one (Corollary 1.6). This fact was proved in [5, Theorem 2.6] for $\dim H \leq 4$.

Finally, we show that the strict inequality in Theorem 1.3 may occur. In particular, we obtain examples of rank 1-homogeneous spaces of nonpositive curvature having planes of zero curvature (hence, they are not symmetric). Moreover, they do not satisfy the visibility axiom.

PRELIMINARIES

Let H be a simply connected homogeneous Riemannian manifold of nonpositive curvature ($K \leq 0$). If γ is a unit geodesic in H, $\operatorname{rank}(\gamma)$ is defined

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to be the dimension of the vector space of parallel Jacobi vector fields on γ. The *rank of* H (denoted by $\operatorname{rank}(H)$) is the minimum of $\operatorname{rank}(\gamma)$ over all unit geodesic γ of H such that $\gamma(0) = p$ for some p in H. This definition was introduced in [3] (H not necessarily a homogeneous space) and it coincides with the usual one if H is a symmetric space (see [5, Preliminaries]). It is clear that $\operatorname{rank}(H) \geq 1$ and $\operatorname{rank}(H) = \dim H$ if and only if H is flat.

Being H homogeneous it admits a simply transitive solvable Lie group of isometries (see [1, Proposition 2.5]) and hence H is isometric to a solvable Lie group G with a left invariant metric of nonpositive curvature.

For G as above, G satisfies the *visibility axiom* if and only if G admits a left invariant metric of negative curvature (see [4, Corollary 2.3]). Hence, if G has sectional curvature $K < 0$, G satisfies the visibility axiom and it also follows, by the definition of rank, that G has rank one.

We recall that if $X, Y, Z \in \mathfrak{g}$, the Lie algebra of G, the Riemannian connection ∇ is given by

$$2\langle \nabla_X Y, Z\rangle = \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle$$

and the sectional curvature K at e, the identity of G, is defined by

$$|X \wedge Y|^2 K(X, Y) = \langle R(X, Y)Y, X\rangle$$

where $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

In general, in a Lie group G, if for any $g \in G$, L_g and R_g denote the left and right translations respectively and $I_g = L_g \circ R_{g^{-1}}$, then the adjoint representation of G defined by $\operatorname{Ad}(g) = (dI_g)_e$ satisfies $I_g(\exp X) = \exp(\operatorname{Ad}(g)X)$ and $\operatorname{Ad}(\exp X) = \operatorname{Exp}(\operatorname{ad}_X)$ for every X in $\mathfrak{g}$, where $\exp\colon \mathfrak{g} \to G$ is the exponential map of G and Exp denote the exponential map in $\mathfrak{gl}(\mathfrak{g})$.

1. The rank of G

Let G be a solvable and simply connected Lie group G with a left invariant metric of nonpositive curvature. If $\mathfrak{g}$ is the Lie algebra of G, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{a}$ where $\mathfrak{a}$, the orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]$ with respect to the metric, is an abelian subalgebra of $\mathfrak{g}$. (see [1, Theorem 5.2].)

If $\mathfrak{g}'^c$ is the complexification of $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ then we have a decomposition in direct sum, $\mathfrak{g}'^c = \sum_\lambda \mathfrak{g}'^c_\lambda$, where $\mathfrak{g}'^c_\lambda = \{U \in \mathfrak{g}'^c \colon (\operatorname{ad}_H - \lambda(H)I)^k \cdot U = 0$ for some $k \geq 1$ and for all $H \in \mathfrak{a}\}$ is the associated root space for the root $\lambda \in (\mathfrak{a}^*)^c$ under the abelian action of $\mathfrak{a}$ on $\mathfrak{g}'$. If $\lambda = \alpha \pm i\beta$ is a root of $\mathfrak{a}$ in $\mathfrak{g}'$ (that is $\mathfrak{g}'^c_\lambda \neq 0$), the generalized root space is defined by $\mathfrak{g}'_{\alpha,\beta} = \mathfrak{g}'_{\alpha,-\beta} = \mathfrak{g}' \cap (\mathfrak{g}'^c_\lambda \oplus \mathfrak{g}'^c_{\bar\lambda})$ and $\mathfrak{g}'$ is the direct sum of the generalized root spaces $\mathfrak{g}'_{\alpha,\beta}$.

We assume that G has no de Rham flat factor; it follows from [2, Theorem 4.6] that the factors $\mathfrak{g}_0 = \sum \mathfrak{g}_{0,\beta}$ and $\mathfrak{a}_0 = \{H \in \mathfrak{a} \colon \alpha(H) = 0$ for all roots $\alpha + i\beta\}$ are zero. Then $\mathfrak{g}' = \sum_{\alpha\neq 0} \mathfrak{g}'_{\alpha,\beta}$.

The following lemma is the key for the proof of Theorem 1.3.

Lemma 1.1. *If $H \in \mathfrak{a}$ satisfies $\alpha(H) > 0$ for $\alpha + i\beta$ a root then*

$$\lim_{t \to +\infty} \operatorname{Exp}(-t\,\mathrm{ad}_H)X = 0 \quad \textit{for all } X \in \mathfrak{g}'_{\alpha,\beta}.$$

Proof. Let $\lambda = \alpha + i\beta$ be a root of $\mathfrak{a}$ in $\mathfrak{g}'$. By definition of $\mathfrak{g}'^{c}_{\lambda}$, $N = (\mathrm{ad}_H - \lambda(H)I)|_{\mathfrak{g}'^{c}_{\lambda}}$ is a nilpotent operator on $\mathfrak{g}'^{c}_{\lambda}$. Then $\mathrm{ad}_H|_{\mathfrak{g}'^{c}_{\lambda}} = \lambda(H)I + N$ and $\operatorname{Exp}(-t\,\mathrm{ad}_H)|_{\mathfrak{g}'^{c}_{\lambda}} = e^{-t\lambda(H)}\operatorname{Exp}(-tN)$; since $|e^{-it\beta(H)}| = 1$ it follows that $\lim_{t\to+\infty}\operatorname{Exp}(-t\,\mathrm{ad}_H)|_{\mathfrak{g}'^{c}_{\lambda}} = 0$ if and only if $\lim_{t\to+\infty} e^{-t\alpha(H)}\operatorname{Exp}(-tN) = 0$ in $Gl(\mathfrak{g}'^{c}_{\lambda})$. We compute this limit in each matricial coordinate (ij). Since N is nilpotent,

$$\operatorname{Exp}(-tN) = \sum_{k=0}^{s} (-1)^k \frac{t^k}{k!} N^k \, (N^{s+1} = 0)$$

and

$$\operatorname{Exp}(-tN)_{ij} = \sum_{k=0}^{s} (-1)^k \frac{t^k}{k!} (N^k)_{ij} = P^s_{ij}(t)$$

is a polynomial in t of degree $s \geq 0$. Then, $\lim_{t\to+\infty} e^{-t\alpha(H)}(\operatorname{Exp}(-tN))_{ij} = \lim_{t\to+\infty} e^{-t\alpha(H)} P^s_{ij}(t) = 0$ since $\alpha(H) > 0$.

Hence, $\lim_{t\to+\infty}\operatorname{Exp}(-t\,\mathrm{ad}_H)U = 0$ for all $U \in \mathfrak{g}'^{c}_{\lambda}$ such that $\lambda = \alpha + i\beta$ is a root and consequently, $\lim_{t\to+\infty}\operatorname{Exp}(-t\,\mathrm{ad}_H)X = 0$ for all $X \in \mathfrak{g}'_{\alpha,\beta}$.

A Jacobi vector field J on a geodesic γ is said to be *stable* if there exists a constant $c > 0$ such that $|J(t)| \leq c$ for all $t \geq 0$. We recall that if γ is a geodesic of G, for every tangent vector v at $\gamma(0)$ there exists a unique stable Jacobi vector field J on γ such that $J(0) = v$ (see [6, Lemma 2.2]). It is obvious that every parallel vector field on a geodesic γ is stable.

Lemma 1.2. *Let $H \in \mathfrak{a}$ be such that $\alpha(H) > 0$ for all roots $\alpha + i\beta$ (such an H exists by* [1, Proposition 5.6] *since G has no flat factor). Then the stable Jacobi vector field J on the geodesic $\gamma_H(t) = \exp tH$ with $J(0) = X \in \mathfrak{g}'$ is given by $J(t) = \widetilde{X}_{\exp tH}$, where $\widetilde{X}$ is the right invariant vector field on G such that $\widetilde{X}_e = X$.*

Proof. It follows from Lemma 1.1 since $\widetilde{X}$ is a Jacobi field on γ_H and $|\widetilde{X}_{\exp tH}| = |\mathrm{Ad}(\exp -tH)X| = |\operatorname{Exp}(-t\,\mathrm{ad}_H)X|$.

Theorem 1.3. *Let G be a solvable and simply connected Lie group with a left invariant metric of nonpositive curvature. If G has no de Rham flat factor then $1 \leq \operatorname{rank}(G) \leq \dim \mathfrak{a}$.*

Proof. Let $\mathfrak{a}' = \{H \in \mathfrak{a} : \alpha(H) > 0 \text{ for all roots } \alpha + i\beta\}$. We will show that if $H \in \mathfrak{a}'$ then there is no parallel Jacobi vector field J on γ_H with $J(0) \neq 0$ in $\mathfrak{g}'$. In fact, if such a J exists, J is a stable vector field on γ_H with $J(0) = X \in \mathfrak{g}'$ and from Lemma 1.2, $J(t) = \widetilde{X}_{\exp tH}$ for all $t \in \mathbf{R}$. This is a contradiction since $\lim_{t\to+\infty}|\widetilde{X}_{\exp tH}| = 0$ and $|J(t)| = |X|$ for all $t \in \mathbf{R}$.

Therefore, J is a parallel vector field on γ_H if and only if $J(t) = (dL_{\exp tH})_e Z$ with $Z \in \mathfrak{a}$ $(\nabla_H Z = 0)$; hence the dimension of the space of parallel Jacobi vector fields on γ_H equals $\dim \mathfrak{a}$ and consequently $\operatorname{rank}(\gamma_H) = \dim \mathfrak{a}$ for all $H \in \mathfrak{a}'$. Hence, $\operatorname{rank}(G) \leq \dim \mathfrak{a}$.

We note that if G admits de Rham flat factor then $\operatorname{rank}(G) \leq \dim \mathfrak{a} + \dim \mathfrak{g}_0$ (see [2, Theorem 4.6]).

Remark 1.4. This bound for $\operatorname{rank}(G)$ is the best possible since in the symmetric case $(\nabla R = 0)$, $\operatorname{rank}(G)$ coincides with $\dim \mathfrak{a}$. In fact, G being a symmetric space of noncompact type (G has no flat factor) it follows from [7, §6, Chapter V] that $\operatorname{rank}(G)$ is the maximal dimension of a flat Euclidean isometrically imbedded in G as a complete totally geodesic submanifold. Consequently $\operatorname{rank}(G) \geq \dim \mathfrak{a}$, since $\exp(\mathfrak{a})$ satisfies the above conditions (see [4, §2]). Hence, Theorem 1.3 implies that $\operatorname{rank}(G) = \dim \mathfrak{a}$.

Theorem 1.5. *Let G be a solvable and simply connected Lie group with a left invariant metric of nonpositive curvature. If G satisfies the visibility axiom then G has rank one.*

Proof. It is a direct corollary of Theorem 1.3, since if G satisfies the visibility axiom then G has no flat factor and $\dim \mathfrak{a} = 1$ (see [4, Theorem 2.3]).

Corollary 1.6. *If H is a Riemannian simply connected homogeneous space of nonpositive curvature satisfying the visibility axiom then H has rank one. In particular, if $H = G/T$ admits a G-invariant metric of negative curvature, H has rank one.*

Proof. The first assertion is immediate and the last one follows from [4, Theorem 2.1].

2. Example

In this section, we exhibit a Lie group G with a left invariant metric of nonpositive curvature such that $\operatorname{rank}(G) = 1$ and $\dim \mathfrak{a} = 2$, thus showing that strict inequality in Theorem 1.3 occurs. In this example the commutator subalgebra $\mathfrak{g}'$ is not abelian, in contrast to the case presented in [5, Example 3.3].

First we give a formula for the sectional curvature for a special case.

Lemma 2.1. *Let $\mathfrak{g}$ be a solvable Lie algebra with an inner product $\langle\ ,\ \rangle$ such that $\mathfrak{a}$, the orthogonal complement of $\mathfrak{g}'$ is abelian. If $\operatorname{ad}_{H|_{\mathfrak{g}'}}$ is symmetric with respect to $\langle\ ,\ \rangle$ for all $H \in \mathfrak{a}$, then*

$$\begin{aligned}
&\langle R(X+H, Y+T)(Y+T), X+H\rangle \\
&\quad = \langle R(X,Y)Y, X\rangle - \|[H,Y]-[T,X]\|^2 - \langle [H,Y]-[T,X], [X,Y]\rangle \\
&\qquad + \langle [[H,Y]-[T,X], X], Y\rangle - \langle [[H,Y]-[T,X], Y], X\rangle
\end{aligned}$$

for all $X, Y \in \mathfrak{g}'$, $H, T \in \mathfrak{a}$.

Proof. Let $X, Y \in \mathfrak{g}'$, $H, T \in \mathfrak{a}$. Since $R(T, H) = 0$ and $\nabla_H = 0$ for all $H, T \in \mathfrak{a}$, by the definition of R we have,

$$\begin{aligned}\langle R(X+H, Y+T)(Y+T), X+H\rangle &= \langle R(X, Y)Y, X\rangle + \langle R(X, T)T, X\rangle \\ &\quad + \langle R(H, Y)Y, H\rangle + 2\langle R(H, Y)Y, X\rangle \\ &\quad + 2\langle R(X, T)Y, X\rangle + 2\langle R(H, Y)T, X\rangle \\ &= \langle R(X, Y)Y, X\rangle - \|[T, X]\|^2 \\ &\quad - \|[H, Y]\|^2 - 2\langle \nabla_{[H,Y]}Y, X\rangle \\ &\quad - 2\langle \nabla_{[T,X]}X, Y\rangle - 2\langle \nabla_{[H,Y]}T, X\rangle.\end{aligned}$$

By using the connection formula,

$$\begin{aligned}2\langle \nabla_{[H,Y]}Y, X\rangle &= \langle [[H, Y], Y], X\rangle - \langle [Y, X], [H, Y]\rangle + \langle [X, [H, Y]], Y\rangle \\ 2\langle \nabla_{[T,X]}X, Y\rangle &= \langle [[T, X], X], Y\rangle - \langle [X, Y], [T, X]\rangle + \langle [Y, [T, X]], X\rangle \\ 2\langle \nabla_{[H,Y]}T, X\rangle &= \langle [[H, Y], T], X\rangle - \langle [T, X], [H, Y]\rangle,\end{aligned}$$

We get,

$$\begin{aligned}&\langle R(X+H, Y+T)(Y+T), X+H\rangle \\ &\quad = \langle R(X, Y)Y, X\rangle - \|[T, X]\|^2 - \|[H, Y]\|^2 + 2\langle [T, X], [H, Y]\rangle \\ &\qquad - \langle [[H, Y] - [T, X], Y], X\rangle \\ &\qquad + \langle [[H, Y] - [T, X], X], Y\rangle - \langle [H, Y] - [T, X], [X, Y]\rangle\end{aligned}$$

and the formula follows.

Example 2.2. Let $\mathfrak{g}$ be the Lie algebra of dimension five that is generated by $\{e_1, e_2 \dots e_5\}$ and Lie bracket

$$\begin{gathered}[e_1, e_2] = \varepsilon e_3, \qquad \varepsilon > 0 \\ [e_1, e_3] = [e_2, e_3] = [e_4, e_5] = 0 \\ [e_4, e_1] = \gamma e_1, \qquad [e_4, e_2] = -\gamma e_2, \qquad [e_4, e_3] = 0 \\ [e_5, e_1] = \alpha e_1, \qquad [e_5, e_2] = \beta e_2, \qquad [e_5, e_3] = (\alpha+\beta)e_3\end{gathered}$$

where $\gamma \neq 0$, $\alpha > 0$, $\beta > 0$ are chosen such that $\gamma^2 - \alpha\beta < 0$.

Note that $\mathfrak{g}'$ is nonabelian, it is spanned by $\{e_1, e_2, e_3\}$ and its center $\mathfrak{z}$ is $\mathbf{R}e_3$.

Let $\langle\ ,\ \rangle$ be the inner product in $\mathfrak{g}$ such that $\langle e_i, e_j\rangle = \delta_{ij}$ $i, j = 1 \dots 5$ and observe that $\mathrm{ad}_{e_4}|_{\mathfrak{g}'}$, $\mathrm{ad}_{e_5}|_{\mathfrak{g}'}$ are symmetric with respect to $\langle\ ,\ \rangle$.

For each $\varepsilon \geq 0$ let $(\mathfrak{g}_\varepsilon, \langle\ ,\ \rangle)$ denote the Lie algebra with the same inner product as the one in $\mathfrak{g}$ and the same Lie bracket except for $[e_1, e_2]_\varepsilon = \varepsilon e_3$. We note that $\mathfrak{g}' = \mathfrak{g}'_\varepsilon = \mathfrak{g}'_0 (\mathfrak{g} = \mathfrak{g}_\varepsilon = \mathfrak{g}_0)$ and $\mathfrak{a} = \mathfrak{a}_\varepsilon$ as vector spaces.

By a straightforward computation, using the connection and the curvature formulas we get

(1) $\nabla_{e_1} e_1 = \gamma e_4 + \alpha e_5$, $\nabla_{e_1} e_2 = \frac{1}{2}\varepsilon e_3$, $\nabla_{e_1} e_3 = -\frac{1}{2}\varepsilon e_2$
$\nabla_{e_2} e_2 = -\gamma e_4 + \beta e_5$, $\nabla_{e_2} e_3 = \frac{1}{2}\varepsilon e_1$, $\nabla_{e_3} e_3 = (\alpha + \beta) e_5$
$\nabla_{e_1} e_4 = -\gamma e_1$, $\nabla_{e_1} e_5 = -\alpha e_1$, $\nabla_{e_2} e_4 = \gamma e_2$
$\nabla_{e_2} e_5 = -\beta e_2$, $\nabla_{e_3} e_4 = 0$, $\nabla_{e_3} e_5 = -(\alpha + \beta) e_3$

(2)
$$K_\varepsilon(e_1, e_2) = -\tfrac{3}{4}\varepsilon^2 + \gamma^2 - \alpha\beta$$
$$K_\varepsilon(e_2, e_3) = \tfrac{1}{4}\varepsilon^2 - \beta(\alpha + \beta)$$
$$K_\varepsilon(e_1, e_3) = \tfrac{1}{4}\varepsilon^2 - \alpha(\alpha + \beta)$$

(3) If $X = ae_1 + be_2$ and Y in $z^\perp$ (the orthogonal complement of z in $\mathscr{g}'$) is independent with X, up to a positive constant,

$$K_\varepsilon(X, Y + ce_3) = |X \wedge Y|^2 K_\varepsilon(e_1, e_2) + c^2[a^2 K_\varepsilon(e_1, e_3) + b^2 K_\varepsilon(e_2, e_3)].$$

Now, if $X, Y \in \mathscr{g}'$, $H, T \in \mathscr{a}$ and $\{X + H, Y + T\}$ are orthonormal vectors in $\mathscr{g}$, it follows from Lemma 2.1 that

(4) $K_\varepsilon(X+H, Y+T) = \langle R_\varepsilon(X, Y)Y, X\rangle - |[H, Y] - [T, X]|^2 + \varepsilon f(X, Y, H, T)$

where R_ε is the curvature tensor for $(\mathscr{g}_\varepsilon, \langle\ ,\ \rangle)$ and f is a continuous function of X, Y, H, T. Moreover, if $G_{5,2}(\mathscr{g})$ is the Grassmann manifold of 2-planes of $\mathscr{g}$, the curvature function $(\varepsilon, \pi) \to K_\varepsilon(\pi)$ is uniformly continuous for $0 \le \varepsilon \le 1$, $\pi \in G_{5,2}(\mathscr{g})$. In fact; since any 2-plane $\pi \subset \mathscr{g}$ is spanned by orthonormal vectors $\{X+H, Y+T\}$ with $X \in z^\perp$, by using (3) and (4) it is an easy computation to show that $K_\varepsilon(\pi)$ is continuous; $G_{5,2}(\mathscr{g})$ being compact, the assertion follows.

As $\varepsilon \to 0$, $(\mathscr{g}_\varepsilon, \langle\ ,\ \rangle)$ tends to a well defined limit algebra $(\mathscr{g}_0, \langle\ ,\ \rangle)$ such that $\mathscr{g}_0'$ is abelian. Then,

(i) If $X, Y \in \mathscr{g}_0'$, $H, T \in \mathscr{a}$ and $\{X + H, Y + T\}$ are orthonormal vectors,

$$K_0(X + H, Y + T) = \langle R_0(X, Y)Y, X\rangle - |[H, Y] - [T, X]|^2.$$

(ii) If $X = ae_1 + be_2$ and $Y \in z^\perp$ is independent with X, by the formula given in (3), up to a positive constant,

$$K_0(X, Y + ce_3) = |X \wedge Y|^2(\gamma^2 - \alpha\beta) - (\alpha + \beta)c^2(a^2\alpha + b^2\beta) < 0.$$

In particular, $(\mathscr{g}_0, \langle\ ,\ \rangle)$ has sectional curvature $K_0 \le 0$.

(iii) Since any 2-plane $\pi \subset \mathscr{g}_0'$ contains a vector in $z^\perp$, we have that there exists a number $r > 0$ such that $K_0(\pi) < -r$ for all 2-plane $\pi \subset \mathscr{g}_0'$. (The curvature function K_0 is continuous and negative on the Grassmann manifold of 2-planes in $\mathscr{g}_0'$ which is compact.)

(iv) From the uniform continuity stated above, there exist $\varepsilon_0 > 0$ such that, if $\varepsilon \le \varepsilon_0$ then

$$K_\varepsilon(\pi) < K_0(\pi) + r \quad \text{for all } 2\text{-plane } \pi \subset \mathscr{g}.$$

Now, let π be a 2-plane in $\mathscr{g}$ spanned by orthonormal vectors $\{X + H, Y + T\}$ with $X, Y \in \mathscr{g}'$, $H, T \in \mathscr{a}$. First we assume that X, Y are independent. Hence, by using (i) and (iii),

$$K_0(\pi) = |X \wedge Y|^2 K_0(X, Y) - |[H, Y] - [T, X]|^2 \le K_0(X, Y) < -r.$$

Consequently, (iv) implies that,

(v) $$K_\varepsilon(\pi) < K_0(\pi) + r \le K_0(X, Y) + r < 0 \quad \text{if } \varepsilon \le \varepsilon_0.$$

If Y is a multiple of X, $K_\varepsilon(\pi) = -|[H, Y] - [T, X]|^2 \le 0$ (the other terms in (4) are zero).

Hence, if $\varepsilon \le \varepsilon_0$ $(\mathscr{g}_\varepsilon, \langle\ ,\ \rangle)$ has sectional curvature $K_\varepsilon \le 0$. Moreover, if $X = ae_1 + be_2$ with $a \ne 0$ $b \ne 0$ is a unit vector, then for $Y \in \mathscr{g}'$, $T \in \mathscr{a}$ such that $\{X, Y + T\}$ are orthonormal we have, if $Y \ne 0$, $K_\varepsilon(X, Y + T) < 0$ ((v)) if $T \ne 0$, $K_\varepsilon(X, T) = -|[X, T]|^2 < 0$. In fact, $[X, T] = a\langle T, \gamma e_4 + \alpha e_5\rangle e_1 + b\langle T, -\gamma e_4 + \beta e_5\rangle e_2$ is nonzero, since $a \ne 0$, $b \ne 0$ and $\gamma e_4 + \alpha e_5$, $-\gamma e_4 + \beta e_5$ are linearly independent in $\mathscr{a}$.

Hence, if $\varepsilon \le \varepsilon_0$ and G_ε is the simply connected Lie group associated to $(\mathscr{g}_\varepsilon, \langle\ ,\ \rangle)$, the geodesic γ in G_ε with $\gamma'(0) = X$ has rank one; therefore, $\operatorname{rank}(G_\varepsilon) = 1$.

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