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APPROXIMATING THE INVARIANT DENSITIES OF TRANSFORMATIONS WITH INFINITELY MANY PIECES ON THE INTERVAL

P. GÓRA AND A. BOYARSKY

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ABSTRACT. Let $I = [0, 1]$ and $\tau: I \to I$ be a piecewise continuous, expanding transformation with infinitely many pieces of monotonicity. We construct a sequence of transformations $\{\tau_n\}$, each having a finite partition, such that their invariant densities converge in L_1 to the invariant density of τ.

1. INTRODUCTION

Let $I = [0, 1]$ and let $\tau: I \to I$ be a piecewise continuous, expanding transformation with finite of infinitely many pieces of monotonicity. When τ admits an absolutely continuous invariant measure (acim) μ, with density function f, we are interested in a method for approximating f. If τ has a finite number of pieces, results are available [3, 5]. In [4], it is shown that piecewise linear Markov maps $\{\tau_n\}$ can be constructed which approach τ uniformly and whose densities $\{f_n\}$ converge in L_1 to f. This is a strong result which cannot be derived from [3] since the essential inequality there involves a constant which depends inversely on the minimum length of the partition.

There are important transformations which have an infinite number of pieces, for example, the Gauss transformation $\tau(x) = 1/x \pmod 1$. Also, first return maps for any map τ give rise naturally to transformations with a countable number of pieces [2]. There are no known approximation theorems that apply to such maps. However, the method of this note will allow the approximation of the density for such τ.

Further motivation for approximating transformations with infinite pieces comes from computer implementations of dynamical systems. Suppose τ has a countable number of pieces. Then any computer representation, $\hat{\tau}$, of τ will necessarily have only a finite number of pieces. This invites the question: will the absolutely continuous measure associated with $\hat{\tau}$ be close to the acim of τ? This problem was studied in [7] in the case when τ has a finite partition. With

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the results of this note, the invariant density of a map with infinite number of pieces can be approximated by the invariant density of a map with finite number of pieces, τ_n, and by the results of [7], the computer representation of τ_n, $\hat{\tau}_n$, has associated with it an absolutely continuous measure which approximates the invariant density of τ_n. This result is of interest since it proves that histograms of computer orbits $\{\hat{\tau}_n^i(x)\}_{i=1}^{\infty}$ can be used to approximate invariant density of the map τ with an infinite number of pieces.

Using a key lemma from [1], a compactness result analogous to that proved in [4] is established in §2 for families of transformations with a countable number of pieces. This is used to prove the following approximation theorem: the invariant density of a piecewise continuous, expanding transformation with a countable number of pieces can be approximated by the densities $\{f_n\}$, invariant under $\{\tau_n\}$, where τ_n is a finite approximation of τ.

In §3, some examples are presented and an application to linear algebra is suggested. In §4, we discuss the application of the main result to first return maps of nonexpanding transformations.

2. Main result

Definition 1. Let τ_n, τ be maps from $I = [0, 1]$ into itself. We say τ_n converges to τ almost uniformly if given any $\varepsilon > 0$ there exist a measurable set $A_\varepsilon \subset I$, $m(A)_\varepsilon > 1 - \varepsilon$, such that $\tau_n \to \tau$ uniformly on A_ε.

The following lemma will be useful in the sequel.

Lemma 1. *Let $\tau_n \to \tau$ almost uniformly. Let f_n be a fixed point of $P_n = P_{\tau_n}$. If $f_n \to f$ weakly in L_1, then $P_\tau f = f$.*

Proof. We shall prove that the measures $f\,dm$ and $(P_\tau f)\,dm$ are equal. To do this it suffices to show that for any $g \in C(I)$: $\int g(f - Pf_\tau)\,dm = 0$. We have

$$\begin{aligned}\left|\int g(f - P_\tau f)\,dm\right| \le \left|\int g(f - f_n)\,dm\right| + \left|\int g(f_n - P_n f_n)\,dm\right| \\ + \left|\int g(P_n f_n - P_\tau f_n)\,dm\right| + \left|\int g(P_\tau f_n - P_\tau f)\,dm\right|.\end{aligned} \tag{1}$$

The first term goes to 0 as $n \to \infty$ since $f_n \to f$ weakly in L_1. The second term is 0 since $P_n f_n = f_n$. Since

$$\int g(P_\tau f_n - P_\tau f)\,dm = \int (g \circ \tau)(f_n - f)\,dm,$$

and $g \circ \tau$ is bounded, using the continuity of g, the weak convergence of f_n to f implies that the fourth term goes to 0 as $n \to \infty$. It remains only to consider the third term. Since $f_n \to f$ weakly as $n \to \infty$, the sequence $\{f_n\}$ is uniformly integrable. Thus, given $\delta > 0$ there exist $\varepsilon > 0$ such that

$$\int_B |f_n|\,dm < \delta \tag{2}$$

for all $n \geq 1$, where $m(B) < \varepsilon$. Since $\tau_n \to \tau$ almost uniformly, $\tau_n \to \tau$ uniformly on a set A_ε, where $m(A_\varepsilon) > 1-\varepsilon$. Returning to the third term of (1), we can write

$$\begin{aligned}
&\left|\int g(P_n f_n - P_\tau f_n)\,dm\right| \\
&\le \left|\int_{A_\varepsilon}(g\circ\tau_n - g\circ\tau)f_n\,dm\right| + \left|\int_{A_\varepsilon^c}(g\circ\tau_n - g\circ\tau)f_n\,dm\right| \\
&\le \sup_{x\in A_\varepsilon}|g\circ\tau_n(x) - g\circ\tau(x)|\int|f_n|\,dm + 2\sup|g|\int_{A_\varepsilon^c}|f_n|\,dm \\
&\le \omega_g\left(\sup_{x\in A_\varepsilon}\tau_n(x) - \tau(x)|\right)\int|f_n|\,dm + 2\sup|g|\int_{A_\varepsilon^c}|f_n|\,dm
\end{aligned} \tag{3}$$

where ω_g is the modulus of continuity of g. Since $\tau_n \to \tau$ uniformly on A_ε and $\int|f_n|\,dm$ are uniformly bounded, the first term in (3) tends to 0 as $n\to\infty$. Since $m(A_\varepsilon^c)\le\varepsilon$, it follows from (2) that $\int_{A_\varepsilon^c}|f_n|\,dm < \delta$, for all n. Hence, given any $\delta > 0$

$$\left|\int g(P_n f_n - P_\tau f_n)\,dm\right| < \omega_g\left(\sup_{x\in A_\varepsilon}|\tau_n(x)-\tau(x)|\right) + K\delta$$

where $K = 2\sup|g|$. Therefore, the third term of (1) can be made arbitrarily small by the proper choice of ε and n, $n\to\infty$, and the result is proved. ■

Definition 2. We say that the transformation $\tau: I\to I$ is countably piecewise expanding if (i) there exists a countable set $S\subset I$ such that for any connected component $J\subset I\backslash S$, $\tau_{|J}$ is a monotonic C^1-function satisfying: $|\tau'|\ge\lambda>2$, (ii) we assume that S has a finite number of limit points $L=\{s_i\}_{i=1}^q$, and that $V_I g = W < +\infty$, where

$$g(x) = \begin{cases} 0, & x\in S \\ |1/\tau'(x)|, & x\notin S. \end{cases}$$

Let τ be a countably piecewise expanding transformation. We shall now describe a procedure for constructing a sequence of transformations $\{\tau_n\}$ which approximate τ. Let $\delta>0$ be smaller than $(\frac{1}{3}\min\{|s_i - s_j|: 1\le i, j\le q, i\ne j\}$. For any point s_i, $i=1,\dots,q$, we define one or two intervals $U_i(\delta)$, as follows: $U_i(\delta) = (s_i, s_i+\delta)$ if s_i is a limit point of S from the right; $U_i(\delta) = (s_i-\delta, s_i)$ if s_i is a limit point of S from the left, or both such intervals (with different indices) if s_i is a limit point of S from both sides.

Let $\varepsilon_0 = \frac{1}{2} - 1/\lambda > 0$. Consider the intervals $U_i(\delta)$, $i=1,\dots,\bar q$, where $\bar q \ge q$ since some of the limit points may be limit points from both sides. Let $H(\delta) = \bigcup_{i=1}^{\bar q} U_i(\delta)$. Since $V_I g$ is finite, we can choose δ_0 so small that $\sup_{H(\delta_0)} g < \varepsilon_0/5$. Now let $\{\delta_n\}_{n=1}^\infty$ be any decreasing sequence of real numbers converging to 0, $\delta_1\le\delta_0$.

We define the approximation τ_n, $n=1,2,\dots$, to τ as follows: a) $\tau_{n|_{U_i(\delta_n)}}$ is linear with $|\tau_n'|\ge\lambda$, $i=1,\dots,\bar q$;

b) $\tau_{n|_{I\setminus H(\delta_0)}}$ is identical with τ. We define

$$g_n(x) = \begin{cases} 0, & x \in S\setminus H(\delta_0); \\ 1/|\tau_n'|, & \text{elsewhere}. \end{cases}$$

By the main result in [1], it follows that any τ_n has an absolutely continuous invariant measure μ_n. Let f_n be the density of this absolutely continuous invariant measure.

Theorem 1. *The set* $\{f_n\}_{n=1}^{\infty}$ *has uniformly bounded variation and is therefore strongly compact in* L_1.

We shall prove this theorem by means of the following lemma.

Lemma 2. *There exists a partition* R *of* I *into intervals such that for any* $\mathcal{J} \in R$ *and any* $n = 1, 2, \ldots V_{\mathcal{J}} g_n < \frac{1}{2}$.

Proof. The jumps of g do not exceed $\sup g$. Therefore, for any $\varepsilon > 0$ and every $y \in I$ there exists some interval E_y containing y, such that $V_{E_y} g < \sup g + \varepsilon$. Since I is compact, there exists a finite subcover of $\{E_y\}_{y\in I}$, call it R_1, such that for any $\mathcal{J} \in R_1$: $V_{\mathcal{J}} g < \sup g + \varepsilon$. Since $\sup g < 1/\lambda = \frac{1}{2} - \varepsilon_0$, putting $\varepsilon = \varepsilon_0/5$ we get $V_{\mathcal{J}} g < \frac{1}{2} - 4\varepsilon_0/5$, for any $\mathcal{J}$ from R_1.

We now refine R_1 a little as follows; let R_2 be the partition whose endpoints are $\{s_1, \ldots, s_q\} \cup \{\text{endpoints of intervals in } R_1\}$. We now further refine R_2 by adding, if necessary, points close to $s_1, \ldots, s_q$ in such a way that the variation of g over the resulting intervals adjacent to $s_1, \ldots, s_q$ is smaller than $2\varepsilon_0/5$. Call this new partition R.

Now $\sup g_n < 1/\lambda$, $\sup_{H(\delta_n)} g < \varepsilon_0/5$ and $V_{\mathcal{J}} g < 2\varepsilon_0/5$ for any $\mathcal{J} \in R$ such that $\mathcal{J} \cap H(\delta_n) \neq \varnothing$, for n sufficiently large. Thus, we have for any $\mathcal{J} \in R$, $\mathcal{J} \cap H(\delta_n) \neq \varnothing$, and n sufficiently large

$$V_{\mathcal{J}} g_n \leq V_{\mathcal{J}} g + \sup_{\mathcal{J}\cap H(\delta_n)} g + \sup_{\mathcal{J}\cap H(\delta_n)} g_n(x) < \frac{3\varepsilon_0}{5} + \frac{1}{\lambda} < \frac{1}{2}.$$

Lemma 3. *For any function* f *of bounded variation, we have*

$$V_I P_{\tau_n} f \leq a V_I f + D\|f\|_1,$$

where

$$a = \sup_n \left(\sup_{x\in I} g_n(x) + \max_{\mathcal{J}\in R} V_{\mathcal{J}} g_n \right) \leq \frac{1}{\lambda} + \frac{1}{2} < 1$$

and

$$D = \sup_n \left(\max_{\mathcal{J}\in R} (V_{\mathcal{J}} g_n / m(J)) \right) \leq \frac{1}{2 \min_{\mathcal{J}\in R} m(J)}$$

and m *is Lebesgue measure on* I.

Proof. Corollary 3 of [1]. ■

We are now ready to prove Theorem 1. Let f_n be an invariant density of τ_n. The existence of f_n follows from Lemma 3 by standard arguments. Again by Lemma 3, we obtain: $V_I f_n < a V_I f_n + D\|f_n\|_1$. Thus, $V_I f_n \leq D/(1-a)$, $n = 1, 2, \ldots$. This completes the proof of Theorem 1. ■

Theorem 2. *If f is a weak-L_1 limit point of the set of τ_n-invariant densities $\{f_n\}_{n\geq 1}$, then f is a τ-invariant density.*

Proof. A direct consequence of Lemma 1, Theorem 1, and the construction of τ_n. ■

Corollary 1. *If τ has a unique acim with density f, then the τ_n-invariant densities f_n converge to f in L_1.*

3. Examples

(1) It is well known that the Gauss transformation $\tau(x) = 1/x \pmod 1$ has invariant density $f(X) = (1/\ln 2)(1/(1+x))$. The transformation τ is countably piecewise expanding and Corollary 1 applies to it. The approximation transformations $\{\tau_n\}$ are defined as follows:

$$\tau_n(x) = \begin{cases} \tau(x) \pmod 1, & 1/n < x \leq 1 \\ -nx + 1, & 0 \leq x \leq 1/n. \end{cases}$$

The invariant density of τ_n can be computed by using a piecewise linear approximation to τ_n, as is done in [4].

(2) Let τ be the piecewise linear, countably piecewise expanding transformation shown in Figure 1.

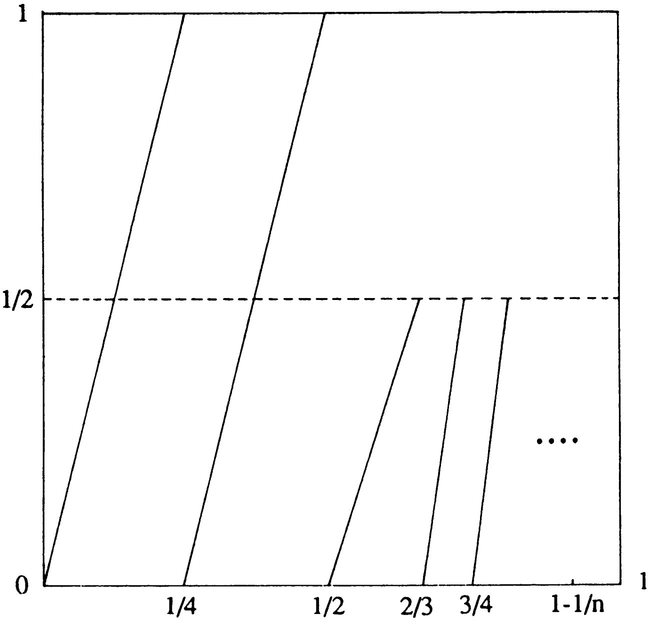

FIGURE 1.

The Frobenius–Perron operator for τ can be represented by the matrix:

$$M = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{4} & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{4} & \cdots \\ \frac{1}{3} & 0 & 0 & \cdots & 0 & \cdots \\ \frac{1}{6} & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & & & & & \\ \frac{2}{n(n+1)} & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & & & & & \end{bmatrix}.$$

Solving for the left eigenvector $\bar{\pi} = (\pi_1, \pi_2, \dots)$, we obtain $\frac{1}{4}\pi_1 + \frac{1}{4}\pi_2 = \pi_2$, which implies that $\pi_2 = \frac{1}{3}\pi_1$. Also, it is easy to see that $\pi_n = \frac{1}{3}\pi_1$, $n \geq 2$. Normalizing, we get:

$$\frac{1}{4}(\pi_1 + \pi_2) + \pi_n \sum_{n=2}^{\infty} \frac{1}{n(n+1)} = 1.$$

On substituting for π_2 and π_n, we obtain $\pi_1 = 2$ and $\pi_n = \frac{2}{3}$, $n \geq 2$. Hence $\bar{\pi} = (2, \frac{2}{3}, \frac{2}{3}, \dots)$ is the unique normalized invariant density of τ. Let us now approximate τ by

$$\tau_N(x) = \begin{cases} \tau(x), & 0 \leq x < 1 - 1/N \\ Nx + 1/(2-N), & 1 - 1/N \leq x \leq 1. \end{cases}$$

Then the invariant density of τ_n is the unique normalized left eigenvector of

$$M_n = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{4} & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{4} & \cdots \\ \frac{1}{3} & 0 & 0 & \cdots & 0 & \cdots \\ \frac{1}{6} & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & & & & & \\ \frac{2}{N} & \frac{2}{N} & 0 & \cdots & 0 & \cdots \end{bmatrix}.$$

By Corollary 1, the sequence of left eigenvectors $\{\bar{\pi}_N\}$ of $\{M_N\}$ converges coordinatewise to $\bar{\pi}$. We remark that Corollary 1 provides a general procedure for truncating certain non-negative infinite-dimensional matrices in such a way that the normalized left eigenvectors of the truncated matrices converge to the normalized left eigenvector of the infinite-dimensional matrix.

4. First return maps

First return maps are often countably piecewise expanding. Consider, for example, $\tau: I \to I$, piecewise monotonic with a finite partition and having the property that at some fixed points $\{x_i\}_{i=1}^{s}$, $|\tau'(x_i)| = 1$. Then τ is a nonexpanding map which has either a σ-finite or finite acim, but the approximation methods of [3,4,5] do not apply. Let $W_1, \dots, W_s$ be neighborhoods of $x_1, \dots, x_s$. Then the first return map, R_W, of τ to the set $W = I \setminus \bigcup_{i=1}^{s} W_i$, is piecewise countable and expanding. R_W can be approximated by the methods of §2. Thus there exists a sequence of piecewise monotonic expanding transformations $\{R_n\}_{n\geq 1}$, each having a finite partition, such that the associated densities $\{g_n\}_{n\geq 1}$ converge to g, the invariant density of R_W.

Let μ_W be the measure induced by g. Then it is easy to show (Lemma 2 of [2]) that for any set $D \subset W$, $\mu_W(D) = \mu_{|_W}(D)$, where $\mu_{|_W}$ is the σ-finite acim of τ restricted to W and normalized on W. Hence the approximating densities $\{g_n\}_{n\geq 1}$, restricted to W, also approximate the density of $\mu_{|_W}$.

Consider the following example [8]:

$$\tau(x) = \begin{cases} \frac{x}{1-x} = \tau_1(x), & 0 \leq x \leq \frac{1}{2} \\ 2x - 1 = \tau_2(x), & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Since $\tau'(0) = 1$, the first return map to the set $G = (\frac{1}{2}, 1)$, say, has a countable number of pieces:

$$R_G(x) = \begin{cases} \tau(x), & x \in (x_0, 1) \\ \tau^{n+1}(x), & x \in (x_n, x_{n+1}), n \geq 1, \end{cases}$$

where $\{x_i\} \subset G$ and $\{y_i\} \subset [0, \frac{1}{2}]$ are defined by

$$y_0 = \frac{1}{2}, \qquad y_{n+1} = \tau_1^{-1}(y_n), \qquad \text{and } x_n = \tau_2^{-1}(y_n), \qquad n \geq 1.$$

R_G can be approximated by the procedure of §2 and the measures of the resulting densities also approximate the normalized measure $\mu_{|_G}$, where μ is the σ-finite measure invariant under τ on the set G.

Remark. An alternate method for approximating the first return map of a non-expanding map is presented in [6].

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DEPARTMENT OF MATHEMATICS, WARSAW UNIVERSITY, WARSAW, POLAND

DEPARTMENT OF MATHEMATICS, CONCORDIA UNIVERSITY, 7141 SHERBROOKE ST. WEST, MONTREAL, CANADA H4B 1R6