PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 4, April 1989

ALGEBRAS OF HOLOMORPHIC FUNCTIONS BETWEEN H^p AND N_*

NOZOMU MOCHIZUKI

(Communicated by Paul S. Muhly)

ABSTRACT. For the algebra N^p, $p > 1$, introduced by Stoll with the notation $(\mathrm{Log}^+ H)^\alpha$ in [5], a characterization of the outer functions will be given, which can be used to derive results analogous to those of N_* [4].

1. THE ALGEBRA N^p

In this section, some introductory remarks will be made. Let U and T denote the unit disk in $\mathbf{C}$ and the unit circle. For $\phi \in L^1(T)$, a holomorphic function $H[\phi]$ is defined by

$$H[\phi](z) = (2\pi)^{-1} \int_0^{2\pi} H(z, e^{it})\phi(e^{it})dt \qquad (z \in U),$$

where $H(z, e^{it}) = (e^{it} + z)(e^{it} - z)^{-1}$. Note that $H = P + iQ$, with P the Poisson kernel. $P[\phi]$ will denote the Poisson integral. We denote by N^p, for $p > 1$, the class of functions f holomorphic in U and satisfying

$$\sup_{0<r<1} \int_0^{2\pi} (\log^+ |f(re^{it})|)^p dt < +\infty.$$

If $f \in N^p$, then $\log(1 + |f^*|) \in L^p(T)$ and

$$(\log(1 + |f(w)|))^p \le P[(\log(1 + |f^*|))^p](w) \qquad (w \in U), \tag{1}$$

where f^* is the boundary function of f on T. Under the metric d_p, defined for $f, g \in N^p$ by

$$d_p(f, g) = \left((2\pi)^{-1} \int_0^{2\pi} (\log(1 + |f^*(e^{it}) - g^*(e^{it})|))^p dt\right)^{1/p},$$

N^p becomes an F-algebra. For $f \in N^p$, (1) implies that

$$\log(1 + |f(w)|) \le 2^{1/p} d_p(f, 0)(1 - |w|)^{-1/p} \qquad (w \in U). \tag{2}$$

Received by the editors July 5, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30H05, 46J15; Secondary 46J20.

It is known that

$$N^q \subset N^p\ (q>p), \quad \bigcup_{p>0} H^p \subset \bigcap_{p>1} N^p, \quad \text{and} \quad \bigcup_{p>1} N^p \subset N_*,$$

where the first containment is proper. To see that the second is proper, let $\phi(e^{it}) = (\log t)^2$ $(t \in (0, 2\pi])$. Then $\phi \in L^p(T)$ for all $p > 1$ and $e^\phi \notin L^p(T)$ for any $p > 0$. Define f by $f(z) = \exp(H[\phi](z))$ $(z \in U)$. Since $(\log^+ |f(z)|)^p \le P[\phi^p](z)$, we have $f \in N^p$ for all $p > 1$. On the other hand, $|f^*| = e^\phi$ a.e. on T implies that $f \notin H^p$, for $p > 0$. Next let $\psi(e^{it}) = t^{-1}(1+|\log t|)^{-2}$ $(t \in (0, 2\pi])$ and define f by $f(z) = \exp(H[\psi](z))$. Since $\psi \in L^1(T)$ and $\log^+|f(z)| = P[\psi](z)$, the uniform integrability of the functions $\{\log^+ |f_r(e^{it})| \,|\, 0 < r < 1\}$ follows, i.e., $f \in N_*$, and $\log^+|f^*| = \psi \notin L^p(T)$, $p > 1$, implies $f \notin N^p$, $p > 1$. Thus the third containment is also proper.

If $f' \in H^p$, $0 < p < 1$, then $f \in H^q$ with $q = p(1-p)^{-1}$ (Hardy-Littlewood, [1]). On the other hand, $f' \in N$ does not imply $f \in N$ (Hayman, [3]). Further, $f' \in N_*$ does not imply $f \in N$ (Yanagihara, [6]). In contrast to H^p, N_*, and N, the class N^p has the following property: If $f' \in N^p$, then $f \in N^p$. If $q > p$, then there exists f such that $f' \in N^p$, yet $f \notin N^q$. The former is easily seen by a maximal function argument [3, p. 183]. To see the latter, let $f(z) = \exp((1-z)^{-\alpha})$ $(z \in U)$ with $q^{-1} < \alpha < p^{-1}$. Since $(1-z)^{-\alpha} \in H^p$, we have $f \in N^p$ and hence $f'(z) = \alpha f(z)(1-z)^{-\alpha-1} \in N^p$. Let $M_\infty(f;r) = \operatorname{Max}\{|f(z)| \,|\, |z| = r\}$. Then $\log^+ M_\infty(f;r) = (1-r)^{-\alpha}$ $(0 < r < 1)$, and hence $(1-r)^{1/q} \log^+ M_\infty(f;r) \to +\infty$ as $r \to 1$. It follows from (2) that $f \notin N^q$.

2. Algebra homomorphisms

By the same argument as in [4], we can prove that if γ is a nontrivial multiplicative linear functional on N^p, then there exists $\lambda \in U$ such that $\gamma(f) = f(\lambda)$ $(f \in N^p)$ and γ is continuous, by (2). This fact will be used to see part (4) of the following Theorem 1.

Let $\Psi: U \to U$ be a holomorphic map. For f holomorphic on U, we define $C_\Psi f$ by

$$(C_\Psi f)(z) = (f \circ \Psi)(z) \qquad (z \in U).$$

Theorem 1. (3) *Let $\Psi: U \to U$ be holomorphic. Then, for $q \ge p$, $C_\Psi: N^q \to N^p$ is a continuous algebra homomorphism.*

(4) *Suppose $\Gamma: N^q \to N^p$ is a nontrivial algebra homomorphism. Then there exists $\Psi: U \to U$, holomorphic, such that $\Gamma f = C_\Psi f$ $(f \in N^q)$. Hence, if $q \ge p$, then Γ is continuous.*

(5) *Suppose $\Gamma: N^q \to N^p$ is an algebra homomorphism onto N^p. Then $p = q$ and Γ is an isomorphism. The map $\Psi: U \to U$, determined by Γ, is a conformal map onto U and $\Gamma^{-1} = C_{\Psi^{-1}}$.*

Proof. (3) Let $f \in N^q$. Then from (1) with $w = \Psi(z)$, (2.5) in [4], and Hölder's inequality we have, for $0 < r < 1$,

$$\begin{aligned}(2\pi)^{-1} &\int_0^{2\pi} (\log(1 + |(f \circ \Psi)(re^{i\theta})|))^p d\theta \\ &\leq \frac{1 + |\Psi(0)|}{1 - |\Psi(0)|} \left((2\pi)^{-1} \int_0^{2\pi} (\log(1 + |f^*(e^{it})|))^q dt \right)^{p/q}.\end{aligned}$$

This shows that $f \circ \Psi \in N^p$ and, at the same time, that $d_p(C_\Psi f, 0) \leq K d_q(f, 0)$ with a constant K independent of f. Thus C_Ψ is continuous. (4) This part is the same as in [4]. (5) Γ is written in the form $\Gamma = C_\Psi$, by (4). $\Psi(U)$ is a nonempty open subset of U, so C_Ψ is one-to-one and $\Gamma^{-1} = C_\Phi$ with a holomorphic map $\Phi: U \to U$. From $\Psi \circ \Phi = \Phi \circ \Psi =$ identity, we see that Ψ is a conformal map of U onto U. Finally, suppose $q < p$ and let $f(z) = \exp((1-z)^{-\alpha})$ with $p^{-1} < \alpha < q^{-1}$. Then $f \notin N^p$ and $f \in N^q$, so $C_\Psi f \in N^p$ by assumption. But we can conclude from (3) that $f = C_\Phi(C_\Psi f)$ belongs to N^p, a contradiction. From $C_\Phi: N^p \to N^q$ we see that $p \geq q$, as well.

3. Outer functions in N^p

It is well known that if $f \in N_*$, then $\log|f^*| \in L^1(T)$. $f \in N^p$ does not imply, however, that $\log|f^*| \in L^p(T)$, while $\log^+|f^*| \in L^p(T)$. Indeed, $f(z) := \exp(H[\psi](z))$ $(z \in U)$ with $\psi(e^{it}) = -t^{-1/p}$ $(t \in (0, 2\pi])$ belongs to H^∞, but $\log|f^*| \notin L^p(T)$. Now let

$$f(z) = a \exp(H[\log \phi](z)) \qquad (z \in U),$$

where $\phi(e^{it}) \geq 0$, $\log \phi \in L^1(T)$, $\log^+ \phi \in L^p(T)$, and $a \in \mathbf{C}$ with $|a| = 1$. We shall call f an *outer function in* N^p. If $f \in N^p$, $f \neq 0$, then f admits the factorization: $f = BSF$, as a function in N_*, where B is the Blaschke product with respect to the zeros of f, S is a singular inner function, and F is an outer function in N_*. Here, since $F = a\exp(H[\log|f^*|])$, F becomes an outer function in N^p. In N_*, f is outer if and only if $f^{-1} \in N_*$. But an outer function in N^p is not necessarily invertible in N^p, as is seen from the example f such that $\log|f^*| \notin L^p(T)$.

Let $f \in N^p$. If there is a sequence $\{f_k\} \subset N^p$ such that $f_k f \to 1$ in N^p as $k \to \infty$, we shall call $\{f_k\}$ an *approximate inverse of* f. This concept characterizes the outer functions in N^p, as follows.

Theorem 2. *Let $f \in N^p$. Then f is outer if and only if f has an approximate inverse. When this is the case, f is approximated by invertible functions in N^p.*

Proof. Suppose first that f is outer in N^p, with $a = 1$: $f(z) = \exp(H[\log \phi](z))$ $(z \in U)$. Let $E_k = \{t \in [0, 2\pi] | \phi(e^{it}) \geq k^{-1}\}$ and $G_k = \{t | \phi(e^{it}) < k^{-1}\}$. Put $\phi_k(e^{it}) = \phi(e^{it})^{-1}$ for $t \in E_k$ and $\phi_k(e^{it}) = 1$ for $t \in G_k$ $(k = 1, 2, \ldots)$.

Then $\log\phi_k \in L^1(T)$ and $\log^+\phi_k \in L^p(T)$, hence $f_k := \exp(H[\log\phi_k])$ belongs to N^p. Put $\psi_k(e^{it}) = 1$ for $t \in E_k$ and $\psi_k(e^{it}) = \phi(e^{it})$ for $t \in G_k$. Then $f_k(z)f(z) = \exp(H[\log\psi_k](z)) = \exp(P[\log\psi_k](z) + iv_k(z))$, where $v_k = Q[\log\psi_k]$. As $r \to 1$, with $z = re^{i\theta}$, we have $P[\log\psi_k]^*(e^{i\theta}) = \log\psi_k(e^{i\theta})$ for a.e. $\theta \in [0, 2\pi]$, and $v_k^*(e^{i\theta})$ also exists for a.e. θ [2, p. 103]. Thus $f_k^*(e^{i\theta})f^*(e^{i\theta}) = \psi_k(e^{i\theta})\exp(iv_k^*(e^{i\theta}))$. Take q, $0 < q < 1$. By Theorem 4.2 in [1], we see that $M_q(v_k; r) \le C_q M_1(P[\log\psi_k]; r) \le C_q\|\log\psi_k\|_1$ $(0 < r < 1)$, where C_q is a constant depending only on q, and hence $\|v_k^*\|_q \le C_q\|\log\psi_k\|_1$, by Fatou's lemma. Since the right side tends to 0 as $k \to \infty$, by the dominated convergence theorem, a subsequence of $\{v_k^*\}$, denoted by the same symbol again, tends to 0 for a.e. $\theta \in [0, 2\pi]$. Hence $f_k^*(e^{i\theta})f^*(e^{i\theta}) \to 1$ as $k \to \infty$, for a.e. θ. Now from $\log(1 + |f_k^* f^* - 1|) \le \log 3$, we conclude that $d_p(f_k f, 1) \to 0$.

Next suppose that $f \in N^p$ and $\{f_k\}$ is an approximate inverse of f. Then we have $f_k(z)f(z) \to 1$ $(z \in U)$ as $k \to \infty$, so $f(z) \ne 0$ $(z \in U)$. Thus the factorization of f is of the form $f = SF$, with S a singular inner function and F outer in N^p. It is enough to see that $S^{-1} \in N^p$, since this implies that S is a constant. Now we have $f_k f S^{-1} = f_k F \in N^p$ and $f_k(z)f(z)S^{-1}(z) \to S^{-1}(z)$ $(z \in U)$ as $k \to \infty$. Since $|(S^{-1})^*| = 1$ a.e. on T, we see that $d_p(f_j f S^{-1}, f_k f S^{-1}) = d_p(f_j f, f_k f) \to 0$ as $j, k \to \infty$. Thus $\{f_k f S^{-1}\}$ converges to some $h \in N^p$, so $f_k(z)f(z)S^{-1}(z) \to h(z)$ $(z \in U)$.

Finally, let f be outer in N^p and define f_k as above. Put $g_k = f_k^{-1}$. Then, since $\log^+\phi_k^{-1} = \log^+\phi \in L^p(T)$, we see that $g_k \in N^p$, i.e., g_k is invertible in N^p. Moreover, $|g_k^*(e^{i\theta})| = |f^*(e^{i\theta})|$ for $\theta \in E_k$ and $|g_k^*(e^{i\theta})| = 1$ for $\theta \in G_k$. Therefore, we have $|g_k^* - f^*| = |g_k^*|\,|f_k^* f^* - 1| \le (|f^*| + 1)|f_k^* f^* - 1|$, the right side tending to 0 as $k \to \infty$, a.e. on T. From $\log(1 + |g_k^* - f^*|) \le \log(2 + 2|f^*|)$, we see that $d_p(g_k, f) \to 0$ as $k \to \infty$.

Remark. Let S be a singular inner function. Then S_r $(0 < r < 1)$ is invertible in N^p, and $S_r \to S$ as $r \to 1$ (Theorem 4.2, [5]). This means that the converse of the second statement of Theorem 2 is not valid.

Corollary. *Let $f \in N^p$. Then fN^p, the ideal generated by f, is dense in N^p if and only if f is outer.*

4. Some ideals in N^p

Theorem 2 above enables us to deduce the following, which are analogues of Theorems 1 and 2 in [4].

Theorem 3. *Let M be a nonzero prime ideal in N^p which is not dense in N^p. Then $M = M_\lambda := \{f \in N^p \mid f(\lambda) = 0\}$ for some $\lambda \in U$. Every closed maximal ideal is of the form M_λ.*

Theorem 4. *Let M be a nonzero closed ideal in N^p. Then there exists a unique (modulo constants) inner function I such that $M = IN^p$.*

Proof. For the proof of Theorem 3, let $f \in M$, $f \neq 0$. Then $f = BSF$, where $F \notin M$ by the above corollary, so we have $BS \in M$. The remainder of the argument is completely analogous to that of [4]. For the proof of Theorem 4, let $f = BSF$, $f \in M$, $f \neq 0$. Take an approximate inverse $\{f_k\}$ of F. Then $f_k f = BS(f_k F) \to BS$ as $k \to \infty$, so we have $BS \in M$ and hence $BS \in M \cap H^1$. The rest is the same as that of [4].

References

1. P. L. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.
2. J. B. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.
3. W. K. Hayman, *On the characteristic of functions meromorphic in the unit disk and of their integrals*, Acta Math. **112** (1964), 181–214.
4. J. W. Roberts and M. Stoll, *Prime and principal ideals in the algebra N^+*, Arch. Math. **27** (1976), 387–393. Correction, ibid. **30** (1978), 672.
5. M. Stoll, *Mean growth and Taylor coefficients of some topological algebras of analytic functions*, Ann. Polon. Math. **35** (1977), 139–158.
6. N. Yanagihara, *On a class of functions and their integrals*, Proc. London Math. Soc. **25** (1972), 550–576.

College of General Education, Tôhoku University, Kawauchi, Sendai 980, Japan