PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 4, April 1989

THE BEHNKE-STEIN THEOREM FOR OPEN RIEMANN SURFACES

R. R. SIMHA

(Communicated by Irwin Kra)

ABSTRACT. Using the Riemann-Roch theorem and the set-topological part of Bishop's special polyhedron lemma, we show that the usual Runge approximation theorem for compact subsets of the Riemann sphere is valid word-for-word on any compact Riemann surface X, with meromorphic functions on X playing the role of rational functions; this result is essentially equivalent to the Behnke-Stein approximation theorem.

1

The Behnke-Stein generalization of the Runge approximation theorem [1], which is the basic tool for many existence questions on open Riemann surfaces, can be stated in several equivalent ways, for instance:

Let X be a Riemann surface, and K a compact subset of X. Then, in order that every holomorphic function in a neighborhood of K be uniformly approximable on K by holomorphic functions on X, it is necessary and sufficient that $X - K$ have no connected component with compact closure in X.

As is well known, the necessity part of the above theorem follows from the sufficiency part (using the theory of compact Riemann surfaces). In this note, we wish to point out how the famous "special polyhedron lemma" of Bishop [2], which is an elementary set-topological result, can be used (together with the Riemann–Roch theorem for compact Riemann surfaces) to give a simple proof of the following theorem.

Theorem 1.1. *Let X be a compact Riemann surface, and $K \subset X$ a compact subset. Let Q be any subset of $X - K$ which contains (precisely) one point q_i from each connected component W_i of $X - K$. Then any holomorphic function on a neighborhood of K can be approximated uniformly on K by meromorphic functions on X whose poles lie in Q.*

Again it is well known and easy to see that Theorem 1.1 implies the sufficiency part of Behnke-Stein. The main fact needed for this implication is the following:

Received by the editors March 24, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30E10, 30F10.
Key words and phrases. Riemann surfaces, Behnke-Stein approximation.

any relatively compact open subset of *any* Riemann surface can also be regarded as an open subset of a *compact* Riemann surface. Observe that, when $X = \mathbb{C}\cup\infty$ is the Riemann sphere and $K \subset \mathbb{C}$, and we take $q_\infty = \infty$ for the component W_∞ of $X-K$ containing ∞, Theorem 1.1 reduces to the usual Runge theorem in $\mathbb{C}$ ([3], p. 176).

The proof of Theorem 1.1 is given in §3; the preliminaries needed, which are easy consequences of the Riemann-Roch theorem, are proved in §2. In §4, we state Bishop's lemma precisely, and indicate how it is especially easy to apply it in our case.

2. Preliminaries

Notation 2.1. For any open subset U of the compact Riemann surface X, $O(U)$ (resp. $M(U)$) denotes the set of holomorphic (resp. meromorphic) functions on U. For any $A \subset X$, $M_A(X)$ denotes the set of $f \in M(X)$ whose poles lie in A.

For any divisor $D = \sum n_i p_i$ on X (finite sum with the p_i distinct), $L(D)$ denotes the sheaf of germs of holomorphic sections of the line bundle $L(D)$ corresponding to D. For any open $U \subset X$, we make the usual identification $H^0(U, L(D)) = \{f \in M(U) : f$ has a zero (resp. pole) of order $\geq -n_i$ (resp. $\leq n_i$) at each $p_i \in U$ with $n_i < 0$ (resp. > 0)$\}$. The $H^0(U, L(D))$ are Frechet spaces in the topology of uniform convergence on compact subsets of U, e.g. they can be regarded as closed subspaces of $O(U - \mathrm{Supp}(D))$, $\mathrm{Supp}(D) = \{p_1, p_2, \dots\}$:

We shall use the Riemann-Roch theorem for compact Riemann surfaces in the following form.

There exists an integer $\delta(X) = \delta$ such that, for any divisor $D = \sum n_i p_i$ on X with $\deg(D) := \sum n_i \geq \delta$, $H^1(X, L(D)) = 0$. (As is well known, we can take $\delta = 2g - 1$, where g is the genus of X.)

Lemma 2.2. *Fix $q_0 \in X$ arbitrarily. Let $p_1, \dots, p_n$ be distinct points in $X - q_0$, and (U_i, z_i) coordinate charts at the p_i. For $1 \leq i \leq n$, let l_i be Laurent polynomials: $l_i(T) = \sum_{j=-d_i}^{d_i} a_{i,j} T^j \in \mathbb{C}[T, T^{-1}]$, d_i integers ≥ 0. Then there exists $f \in M(X)$, with poles at the most at $q_0, p_1, \dots, p_n$, such that, for each i, $f - l_i(Z_i)$ has a zero of order $\geq d_i + 1$ at p_i.*

Proof. Let D be the divisor $d_0 q_0 - \sum(d_i + 1)p_i$, with d_0 so large that $\deg(D) \geq \delta$. We may suppose that the U_i are pairwise disjoint, and regard the $l_i(Z_i)$ as defining a 1-cocycle for $L(D)$ with respect to the covering $(U_1, \dots, U_n, X - \{p_1, \dots, p_n\})$. This cocycle is a coboundary by Riemann-Roch, and the lemma follows from this.

Corollary 2.3. *For every $p \in X - q_0$, there exists $f_p \in M_{p,q_0}(X)$ with a simple pole at p.*

Lemma 2.4. *Let p, f_p be as in 2.3. Let $\alpha: I(= [0, 1] \subset \mathbb{R}) \to X - q_0$ be a path in $X - q_0$. Then there is a continuous map $t \to f_t$ of I into $M(X)$ such that*

(i) $f_0 = f_p$, *and* (ii) $f_t \in M_{\alpha(t),q_0}$ *and has a simple pole at* $\alpha(t)$, *for all* $t \in I$. (*The continuity of* $t \to f_t$ *means that the map* $(t, x) \to f_t(x)$ *of* $I \times X$ *into* $\mathbf{C} \cup \infty$ *is continuous.*)

Proof. Clearly, we may assume that there is a coordinate disc (U, z) with $q_0 \notin U$ and $\alpha(I) \subset U$. Let $V \subset\subset U$ be a smaller disc with $\alpha(I) \subset V$. There is a $\lambda \in \mathbf{C}^*$ such that $f_p - \lambda/(z - z(p))$ is holomorphic in U. Put $\beta(t) = z(\alpha(t))$, and regard $t \to l_t = \lambda(z-\beta(t))^{-1} - \lambda(z-\beta(0))^{-1}$ as a continuous map $I \to H^0(U - \overline{V}, L(\delta q_0)) = O(U - \overline{V})$. By Riemann-Roch, the obvious sequence

$$0 \to H^0(X, L(\delta q_0)) \to H^0(X - \overline{V}, L(\delta q_0)) \oplus O(U) \to O(U - \overline{V}) \to 0$$

is exact. Moreover, $H^0(X, L(\delta q_0))$ is finite dimensional, hence this sequence, regarded as an exact sequence of continuous linear maps of Frechet spaces, splits. (An explicit splitting is easy to give in this special case.) Hence we have continuous maps $t \to \varphi_t$, $t \to \psi_t$ of I into $H^0(X - \overline{V}, L(\delta q_0))$ and $O(U)$ respectively, such that $\varphi_0 = \psi_0 = 0$ and $\varphi_t - \psi_t = l_t$ on $U - \overline{V}$. Setting $g_t = \varphi_t$ on $X - \overline{V}$ and $l_t + \psi_t$ on U, we see that the family $t \to f_p + g_t$, $t \in I$ has the desired properties.

Corollary 2.5. *Let* $\alpha: I \to X$ *be a path with* $\alpha(0) = p$ $(\neq q_0)$ *and* $\alpha(1) = q_0$. *Let* $K \subset X$ *be a compact set with* $K \cap \alpha(I) = \varnothing$. *Then every* $f \in M_{p,q_0}(X)$ *can be approximated uniformly on* K *by elements of* $M_{q_0}(X)$.

Proof. We may assume $\alpha(t) \neq q_0$ for $t < 1$. Pick (e.g. using Lemma 2.2) any nonconstant $f_{q_0} \in M_{q_0}(X)$. Let V be a neighborhood of q_0 such that $\inf\{|f_{q_0}(x)| : x \in V\} > 2\sup\{|f_{q_0}(x)| : x \in K\}$. Choose $t' \in I$ such that $\alpha(t') \in V$. Let (f_t) be the functions constructed in 2.4 for the path $\alpha[0, t']$. Clearly, we can subdivide $[0, t'] : 0 = t_0 < t_1 < \cdots < t_n = t'$ such that $|f_{t_i}(\alpha(t_{i-1}))| > 2M$, $i = 1, \ldots, n$, where $M := \sup\{|f_t(x)| : 0 \le t \le t', x \in K\}$ $(< \infty)$.

Now let $f \in M_{p,q_0}(X)$. Then $f(f_{t_1} - f_{t_1}(p))^m \in M_{\alpha(t_1),q_0}(X)$ for some integer $m > 0$. And $(f_{t_1}(p) - f_{t_1})^{-1} = f_{t_1}(p)^{-1}(1 - f_{t_1}/f_{t_1}(p))^{-1}$ can be approximated on K by a polynomial in f_{t_1} using the geometric series. Repeating this argument $n - 1$ more times, we see that f can be approximated on K by elements of $M_{\alpha(t'),q_0}(X)$. The same argument, repeated this time using f_{q_0}, shows that any $g \in M_{\alpha(t'),q_0}(X)$ can be approximated on K by elements of $M_{q_0}(X)$, and the corollary is proved.

3

We now come to the proof of Theorem 1.1. With the notation of 1.1, let $U \supset K$ be an open set. For each $p \in \partial U$, there exists $f_p \in M_p(X)$ with a pole at p. We may assume that $|f_p| < 1$ on K. Let V_p be a neighborhood of p such that $|f_p| > 2$ on V_p. Finitely many of the V_p, say $V_{p_1}, \ldots, V_{p_m}$, cover

∂U. Thus, with

$$V := \{x \in U : |f_{p_i}(x)| < 1 \text{ for } 1- \leq i \leq m\}$$

we have $K \subset V \subset\subset U$. By the procedure due to Bishop [2] (see §4), we can then find a $\varphi \in M(X)$ which is in fact a polynomial in the f_{p_i}, and an open set $V_0 \subset V$, such that $K \subset V_0$, and $\varphi : V_0 \to \Delta$ $(= \{|z| < 1\} \subset \mathbb{C}\}$ is a *proper map*. Hence it is clear that the following lemma, together with Corollary 2.5, implies Theorem 1.1.

Lemma 3.1. *Let V_0 be any open subset of X, and $\varphi : V_0 \to \Delta$ a proper holomorphic map to the unit disc in $\mathbb{C}$. Let $q_0 \in X - V_0$ be arbitrary. Then any $h \in O(V_0)$ can be approximated uniformly on any compact subset K of V_0 by polynomials in φ and the elements of $M_{q_0}(X)$.*

Proof. Since $\varphi : V_0 \to \Delta$ is a proper map, there exists a discrete subset E of Δ such that $\varphi : V_0 - \varphi^{-1}(E) \to \Delta - E$ is a (possibly disconnected) finite-sheeted (say n-sheeted) covering. Pick some $z_0 \in \Delta - E$: by Lemma 2.2, there exists $\varphi_1 \in M_{q_0}(X)$ taking n distinct values on $\varphi^{-1}(Z_0)$. Let $P(z, T) \in O(\Delta)[T]$ be the monic polynomial of degree n, defined for $z \in \Delta - E$ by $P(z, T) = \prod_{\varphi(p)=z}(T - \varphi_1(p))$. Define $\psi \in O(V_0)$ by $\psi(p) = (\partial P/\partial T)(\varphi(p), \varphi_1(p))$. Then $\psi \not\equiv 0$ on any connected component of V_0, since $\psi \neq 0$ on $\varphi^{-1}(z_0)$.

Now, for any disc $\Delta_1 \subset \Delta$, and $V_1 := \varphi^{-1}(\Delta_1)$, consider, for any $h \in O(V_1)$, the polynomial $Q_h \in O(\Delta)[T]$, defined for $z \in \Delta_1 - E$ by

$$Q_h(z, T) = \sum_{i=1}^{n} h(p_i) \prod_{j \neq i} (T - \varphi_1(p_j)),$$

where $\{p_1, \dots, p_n\} = \varphi^{-1}(z)$. Then $Q_h(\varphi(p), \varphi_1(p)) = h(p)\psi(p)$ for $p \in V_1 - \varphi^{-1}(E)$ by definition, hence for all $p \in V_1$.

Now we can choose $\Delta_1 \subset\subset \Delta$ such that $\varphi(K) \subset \Delta_1$. Then $V_1 \subset\subset V_0$, hence ψ has only finitely many zeros in V_1. Hence, for any given $h \in O(V_1)$, we can find (by Lemma 2.2) a $\varphi_2 \in M_{q_0}(X)$ such that $h - \varphi_2$ is divisible by ψ in $O(V_1)$: say $h - \varphi_2 = \psi h'$. But $\psi(p')h'(p) = Q_{h'}(\varphi(p), \varphi_1(p))$ for the $Q_{h'} \in O(\Delta_1)[T]$ constructed above. Thus, using Taylor expansions to approximate the coefficients of $Q_{h'}$ on $\varphi(K)$, we see that ψh can be approximated on $K \Subset \varphi^{-1}(\varphi(K))$ by polynomials in φ and φ_1. □

4. Bishop's Lemma

We state below the lemma of Bishop used in §3. Since we have not found a reference where the result is proved in the stated form, we first state the two simple facts using which the lemma can be easily proved.

Lemma 4.1 ([2], p. 222). *For $\nu \geq 1$, the set $\{z \in \mathbb{C}: |z^\nu - 1| < \frac{1}{2}\}$ is contained in the union of the ν pairwise disjoint sets*

$$W_{k,\nu} = \left\{ \left(\frac{1}{2}\right)^{1/\nu} < |z| < \left(\frac{3}{2}\right)^{1/\nu}, |\arg z - 2k\pi/\nu| < \pi/\nu \right\}, k = 0, \ldots, \nu-1;$$

the diameter of $W_{k,\nu}$ is $O(\nu^{-1})$ as $\nu \to \infty$.

Lemma 4.2. *Let Y, Z be metric spaces, with Y compact, and $f: Y \to Z$ a continuous map with totally disconnected fibers. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that every connected subset of Y of diameter $\geq \varepsilon$ has f-image of diameter $\geq \delta$.*

The argument of Bishop [2] for his proof of Theorem 2 (p. 221 of [2]) proves exactly the following Lemma 4.3: 4.2 can be used in place of the function theory on p. 223 of [2].

Bishop's Lemma 4.3. *Let U be a locally compact locally arcwise-connected metric space, and $f_i: U \to \mathbb{C}$ continuous functions $1 \leq i \leq m$. Let $V \subset\subset U$ be an open subset such that $f = (f_1, \ldots, f_m)$ maps V properly into the unit polydisc $\{Z_i < 1, 1 \leq i \leq m\}$ in $\mathbb{C}^m$. Let $U_1 = \{x \in U: f_1(X) \neq 0\}$, and suppose the map $g = (f_2/f_1, \ldots, f_m/f_1): U_1 \to \mathbb{C}^{m-1}$ has totally disconnected fibers. Then, for any compact $K \subset V$, there exist $r > 1$ and a positive integer ν such that the set*

$$V' = \{x \in U: |(rf_1(x))^\nu - (rf_j(x))^\nu| < 1, j = 2, \ldots, m\}$$

contains K, and the union V_0 of the connected components of V' meeting K is contained in V. Hence $V_0 \subset\subset U$, and the $m-1$ functions $(rf_1)^\nu - (rf_j)^\nu, j = 2, \ldots, m$ map V_0 properly into the unit polydisc in $\mathbb{C}^{m-1}$.

We conclude with the observation that, in our applications of this lemma, the condition that the map $g: U_1 \to \mathbb{C}^{m-1}$ have discrete fibers is automatically satisfied. This is because the f_i will be holomorphic functions on a connected Riemann surface X' containing U as an open set. Thus g will fail to satisfy the required condition only if $g_j = \lambda_j g_1, \lambda_j \in \mathbb{C}, j \geq 2$. In this case, it is obvious that a suitable constant multiple $\varphi := \lambda g_1$ will already have the desired property that $\varphi: V \to \Delta$ be a proper map. Thus, after at the most $m-1$ applications of Bishop's lemma, we will have V_0 and φ as required in §3.

References

1. H. Behnke and K. Stein, *Entwicklungen analytischer Funktionen auf Riemannschen Flächen*, Math. Ann. **120** (1948), 430–461.
2. E. Bishop, *Mappings of partially analytic spaces*, Amer. J. Math. **83** (1961), 209–242.
3. S. Saks and A. Zygmund, *Analytic functions*, Warsaw, 1952.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India