# THE SPECTRAL AND FREDHOLM THEORY OF EXTENSIONS OF BOUNDED LINEAR OPERATORS 

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#### Abstract

Assume $T$ is a bounded linear operator on some Banach space $Y$, and that $T$ has a bounded extension $\bar{T}$ on another space. In general almost nothing can be said concerning the relationship between the spectral and Fredholm properties of $T$ and $\bar{T}$. However, assuming the special condition that the range of $\bar{T}$ lies in $Y$, it is shown that these properties are essentially the same for $T$ and $\bar{T}$.


## 1. Introduction

Let $X$ be a Banach space, and let $Y$ be a subspace of $X$ which is also a Banach space. Assume the embedding of $Y$ into $X$ is continuous. Let $\mathscr{B}(X)$ denote the algebra of all bounded linear operators on $X$. In this paper we are concerned with the situation where $T \in \mathscr{B}(Y)$ has an extension $\bar{T} \in \mathscr{B}(X)$. In this case it is often of interest to know how the spectral and Fredholm properties of $T$ relate to those of $\bar{T}$. In fact, not much can be said in general, as is apparent from the three examples at the end of this section. There is one general result concerning the spectrum:
I. Assume $Y$ is dense in $X$. It follows from a theorem of S . Grabiner [7, Theorem 2.1] or from a theorem of B. Barnes [2, Theorem 4.5] that every component of $\sigma(T)$ (the spectrum of $T$ ) has nonempty intersection with $\sigma(\bar{T})$, and every component of $\sigma(\bar{T})$ has nonempty intersection with $\sigma(T)$.

Other results are known in special situations:
II. Assume $Y$ has a bounded innerproduct, and $X$ is the Hilbert space completion of $Y$ in the norm determined by the innerproduct. When $T \in$ $\mathscr{B}(Y)$ and $T$ is symmetric with respect to the innerproduct, then P . Lax proved that $T$ has an extension $\bar{T} \in \mathscr{B}(X)$ and $\sigma(\bar{T}) \subseteq \sigma(T)$ [11]. Generalizations of this result, and some related results, are proved in [9, pp. 364-366].

[^0]III. In a situation similar to (II), but assuming only that $T$ is symmetric with respect to a bounded pre-innerproduct on $Y$, then $T$ inherits many of the special properties of the self-adjoint extension $\bar{T}$ provided that $\bar{T}(X) \subseteq Y$. In particular, $T$ has a rich operational calculus defined in terms of the operational calculus of the self-adjoint operator $\bar{T}$. These results are contained in [3].

In order to illustrate the problems inherent in the general situation we give three examples. The notation here is

$$
\begin{gathered}
\omega(S)=\{\lambda \in \mathbf{C}: \lambda-S \text { is not a Fredholm operator }\} \\
W(S)=\{\lambda \in \mathbf{C}: \lambda-S \text { is not a Fredholm operator of index zero }\}
\end{gathered}
$$

Thus, $\omega(S)$ is the usual Fredholm (or essential) spectrum of an operator $S$, while $W(S)$ is the Weyl spectrum of $S$.

Example 1. Let $D=\{z \in \mathbf{C}:|z| \leq 1\}$. Let $Y$ be the disk algebra, the space of all continuous functions on $D$ which are holomorphic on the interior of $D$ [13, p 2]. The complete norm on $Y$ is

$$
\|f\|=\sup \{|f(z)|: z \in D\} \quad(f \in Y)
$$

Consider the norm on $Y$

$$
\|f\|_{0}=\sup \{|f(1 / n)|: n \geq 1\} \quad(f \in Y)
$$

Let $X$ be the completion of $Y$ with respect to this norm. Then $X$ is isometrically isomorphic to the classical Banach space of all complex convergent sequences. Define $T \in \mathscr{B}(Y)$ by

$$
T(f)(z)=z f(z) \quad(f \in Y)
$$

Then $T$ has an extension $\bar{T} \in \mathscr{B}(X)$. Also, $\bar{T}$ is a compact operator on $X$, $\sigma(\bar{T})=\{0\} \cup\{1 / n: n \geq 1\}$, and $\omega(\bar{T})=\{0\}$. For $T$ on the other hand, $\sigma(T)=\omega(T)=D$.
Example 2. Let $Y$ be the Banach space of all sequences $a=\left\{a_{k}\right\}_{k=1}^{\infty}$ such that

$$
\|a\|_{Y}=\sum_{k=1}^{\infty} 2^{k}\left|a_{k}\right|<+\infty
$$

Then $Y$ is a dense subspace of $X=l^{2}$. Let $V$ and $W$ be the unilateral shift and backward shift acting on $Y$, and let $\bar{V}$ and $\bar{W}$ be their extensions to $X$. Set $T=2 I+V+W$. Define

$$
\Omega=\left\{\lambda=x+i y, x, y \in \mathbf{R}:(2 / 5)^{2}(x-2)^{2}+(2 / 3)^{2} y^{2} \leq 1\right\}
$$

In [12, p. 150] J. Nieto shows that $\sigma(T)=W(T)=\Omega$, and $\omega(T)=\partial \Omega$, while $\sigma(\bar{T})=W(\bar{T})=\omega(\bar{T})=\{x \in \mathbf{R}: 0 \leq x \leq 4\}$. Note that $\bar{T}$ is self-adjoint.

Example 3. Define $T$ on $l^{1}$ by $T\left(\left\{a_{k}\right\}\right)=\left\{b_{k}\right\}$ where $b_{k}=\left(a_{k+1}\right) /(k+1)$, $k \geq 1$ ( $T$ is a weighted backward shift). The operator $T$ is compact on $l^{1}$,
and $\sigma(T)=\{0\}$. Now let $X$ be the space of all complex sequences $a=\left\{a_{k}\right\}$ such that

$$
\|a\|_{X}=\sum_{k=1}^{\infty}(1 / k!)\left|a_{k}\right|<+\infty .
$$

Certainly $l^{1}$ can be identified as a subspace of $X$. Also, $\bar{T}\left(\left\{a_{k}\right\}\right)=\left\{b_{k}\right\}$ with $b_{k}$ as above is a bounded extension of $T$ on $X$. Define $V: l^{1} \rightarrow X$ by $V\left(\left\{a_{k}\right\}\right)=\left\{v_{k}\right\}$ where $v_{k}=k!a_{k}, k \geq 1$. Then $V$ is an isometry of $l^{1}$ onto $X$. It is easy to calculate that $V^{-1} \bar{T} V$ is the usual backward shift on $l^{1}$. Thus, $\sigma(\bar{T})=D, W(\bar{T})=D, \omega(\bar{T})=\{\lambda:|\lambda|=1\}$, and $\operatorname{ind}(\lambda-\bar{T})=1$ for $\lambda \in \mathbf{C}$, $|\lambda|<1$.

As seen in Examples 1-3 the spectral and Fredholm properties of $T$ and $\bar{T}$ can be wildly different in general. In this paper we make the special assumption that $\bar{T}(X) \subseteq Y$, and with this assumption it is shown that the spectral and Fredholm properties of $T$ and $\bar{T}$ are essentially the same. This is done in $\S 2$. In $\S 3$ an application to operators on Lebesgue spaces is given.

## 2. The spectral theory of extensions

As before $X$ and $Y$ are Banach spaces with $Y$ continuously embedded in $X$. We assume that $Y \neq X$ throughout. We are concerned with the situation where $T \in \mathscr{B}(Y)$ has an extension $\bar{T} \in \mathscr{B}(X)$. Let $\mathfrak{L}$ be the set of all operators $T \in \mathscr{B}(Y)$ which have an extension $\bar{T} \in \mathscr{B}(X)$ with the property $\bar{T}(X) \subseteq Y$. Note that $\mathfrak{L}$ is a left ideal in $\mathscr{B}(Y)$, a fact that will be exploited in what follows.

In the Fredholm theory of operators in $\mathscr{B}(Y)$, an important role is played by the ideal of finite rank operators in $\mathscr{B}(Y)$. We denote this ideal by $\mathscr{F}(Y)$. An operator in $\mathscr{F}(Y)$ with one-dimensional range has the form $\alpha \otimes y$ where $y \in Y$ and $\alpha \in Y^{\prime}$ ( $Y^{\prime}$ is the dual space of $Y$ ). The operator $\alpha \otimes y$ acts according to the rule

$$
\alpha \otimes y(z)=\alpha(z) y \quad(z \in Y)
$$

For $T \in \mathscr{B}(X)$, let $\mathscr{R}(T)$ be the range of $T$ and $\mathscr{N}(T)$ be the null space of $T$. Let $\operatorname{Inv}(X)$ be the group of invertible operators in $\mathscr{B}(X)$. The set of all Fredholm operators on $X$ is denoted $\Phi(X)$, and $\Phi^{0}(X)$ is the set of all $T \in \Phi(X)$ with index zero. The notation $\operatorname{nul}(T), \operatorname{def}(T)$, and $\operatorname{ind}(T)$ is clear (the nullity, defect, and index of $T$ ). As in the Introduction, $\omega(T)=\{\lambda \in \mathbf{C}: \lambda-T \notin \Phi(X)\}, \quad$ and $\quad W(T)=\left\{\lambda \in \mathbf{C}: \lambda-T \notin \Phi^{0}(X)\right\}$.
Also, let $\sigma^{\prime}(T)=\{\lambda \in \sigma(T): \lambda \neq 0\}$, and $\omega^{\prime}(T)$ and $W^{\prime}(T)$ are defined similarly.

Now we state the main result of the paper.
Theorem 4. Assume $T \in \mathscr{B}(Y)$ and $T$ has an extension $\bar{T} \in \mathscr{B}(X)$ with $\bar{T}(X) \subseteq Y$.
(1) $\sigma^{\prime}(T)=\sigma^{\prime}(\bar{T}), \omega^{\prime}(T)=\omega^{\prime}(\bar{T})$, and $W^{\prime}(T)=W^{\prime}(\bar{T})$. If $\lambda \notin \omega(T)$, $\lambda \neq 0$, then $\operatorname{ind}(\lambda-\bar{T})=\operatorname{ind}(\lambda-T)$.
(2) When $Y$ is a proper dense subspace of $X$, then $\sigma(T)=\sigma(\bar{T}), \omega(T)=$ $\omega(\bar{T})$, and $W(T)=W(\bar{T})$.
(3) When $Y$ is closed in $X$, but $Y$ has no closed complement in $X$, then $\sigma(T)=\sigma(\bar{T}), \omega(T)=\omega(\bar{T})$, and $W(T)=W(\bar{T})$.

The proof of part (1) of the theorem is contained in the next four lemmas. When $T \in \mathscr{B}(Y)$, then $\bar{T}$ always denotes an extension of $T$ with $\bar{T} \in \mathscr{B}(X)$. In addition when $T \in \mathfrak{L}$, then it is assumed that $\bar{T}(X) \subseteq Y$.

It is convenient to label two separate cases:
(C) $Y$ is a closed subspace of $X$;
(D) $Y$ is dense in $X$.

Concerning these cases, we have the following facts:
Note. When (D) holds, then $T \in \mathfrak{L}$ is equivalent to the property

$$
\text { (\#) } \exists J>0 \text { such that }\|T y\|_{Y} \leq J\|y\|_{X} \quad(y \in Y)
$$

When (C) holds, then $\mathscr{F}(Y) \subseteq \mathfrak{L}$. For in this case when $F=\alpha \otimes y \in \mathscr{F}(Y)$, then by the Hahn-Banach Theorem $\alpha$ has an extension $\bar{\alpha} \in X^{\prime}$. Then $\bar{F}=$ $\bar{\alpha} \otimes y \in \mathscr{B}(X), \bar{F}(X) \subseteq Y$, and $\bar{F}(z)=F(z)$ for all $z \in Y$.

Lemma 5. Assume $R$ and $T$ are in $\mathfrak{L}$ and $\lambda \in \mathbf{C}, \lambda \neq 0$.
(1) $\lambda-R \in \operatorname{Inv}(Y) \Leftrightarrow \lambda-\bar{R} \in \operatorname{Inv}(X)$.
(2) Assuming (C) holds, if $\lambda-T \in \Phi^{0}(Y)$, then $\lambda-\bar{T} \in \Phi^{0}(X)$.

Proof. Assume that $R \in \mathfrak{L}$ and that $\lambda \neq 0$. If $(\lambda-\bar{R}) x=0$, then $\lambda x=\bar{R}(x) \in$ $Y$. This proves that $\mathscr{N}(\lambda-\bar{R})=\mathscr{N}(\lambda-R)$. Assume $\lambda-R \in \operatorname{Inv}(Y)$, and fix $z \in X$. Since $\bar{R}(z) \in Y, \exists y \in Y$ such that $(\lambda-R) y=\bar{R}(z)$. Therefore

$$
(\lambda-\bar{R})(z+y)=\lambda z-\bar{R}(z)+(\lambda-R) y=\lambda z
$$

This proves $\mathscr{R}(\lambda-\bar{R})=X$, and since $\mathscr{N}(\lambda-\bar{R})=\mathscr{N}(\lambda-R)=\{0\}, \lambda-\bar{R} \in$ $\operatorname{Inv}(X)$.

Assume $\lambda-\bar{R} \in \operatorname{Inv}(X)$. Fix $y \in Y$. Then $\exists x \in X$ with $(\lambda-\bar{R}) x=y$. Since $\bar{R}(x) \in Y, \lambda x=\bar{R} x+y \in Y$, so $x \in Y$. This proves $\mathscr{R}(\lambda-R)=Y$, so $\lambda-R \in \operatorname{Inv}(Y)$.

Now suppose (C) holds and $\lambda-T \in \Phi^{0}(Y)$. By [1, Theorem 0.2.8] we can write $\lambda-T=S+F$ where $S \in \operatorname{Inv}(Y)$ and $F \in \mathscr{F}(Y)$. As noted above, since (C) holds, $F \in \mathfrak{L}$ and $F$ has an extension $\bar{F} \in \mathscr{F}(X)$. By part (1) $\lambda-\bar{T}-\bar{F} \in \operatorname{Inv}(X)$, and this proves $\lambda-\bar{T} \in \Phi^{0}(X)$.

Lemma 6. Let $T \in \mathfrak{L}$. Assume $\lambda \in \mathbf{C}, \lambda \neq 0$, and $\lambda-\bar{T} \in \Phi(X)$. Then $\lambda-T \in \Phi(Y)$ and $\operatorname{ind}(\lambda-\bar{T}) \leq \operatorname{ind}(\lambda-T)$.
Proof. Just as in Lemma 5, we have $\mathscr{N}(\lambda-T)=\mathscr{N}(\lambda-\bar{T})$. Thus, $\operatorname{nul}(\lambda-T)=$ $\operatorname{nul}(\lambda-\bar{T})$.

Now assume $\left\{y_{1}, \ldots, y_{n}\right\}$ is a linearly independent set in $Y$ with

$$
\operatorname{span}\left\{y_{1}, \ldots, y_{n}\right\} \cap \mathscr{R}(\lambda-T)=\{0\} .
$$

Suppose for some $x \in X$ and $\left\{\lambda_{1}, \ldots, \lambda_{n}\right\} \subset C$, we have

$$
\lambda_{1} y_{1}+\cdots+\lambda_{n} y_{n}=(\lambda-\bar{T}) x
$$

Then $\lambda x=\lambda_{1} y_{1}+\cdots+\lambda_{n} y_{n}+\bar{T} x \in Y$. But this implies $(\lambda-T) x=0$. Thus, $\mathscr{R}(\lambda-\bar{T})$ has codimension at least $n$. It follows that $\mathscr{R}(\lambda-T)$ must have finite codimension in $Y$. By [5, Cor. (3.2.5)] $\mathscr{R}(\lambda-T)$ must be closed, hence $\lambda-T \in \Phi(Y)$. Also the argument shows $\operatorname{def}(\lambda-T) \leq \operatorname{def}(\lambda-\bar{T})$. Thus, $\operatorname{ind}(\lambda-\bar{T}) \leq \operatorname{ind}(\lambda-T)$.

Lemma 7. Assume (C) holds. Suppose $T \in \mathfrak{L}, \lambda \in \mathbf{C}, \lambda \neq 0$, and $\lambda-T \in \Phi(Y)$. Then $\lambda-\bar{T} \in \Phi(X)$ and $\operatorname{ind}(\lambda-T)=\operatorname{ind}(\lambda-\bar{T})$.
Proof. We may assume $\lambda=1$. Since $I-T \in \Phi(Y), \exists S \in \mathscr{B}(Y)$ and $\exists F$, $G \in \mathscr{F}(Y)$ such that

$$
(I-T)(I-S)=I-F \quad \text { and } \quad(I-S)(I-T)=I-G
$$

Because (C) holds, $G \in \mathscr{F}(Y) \subseteq \mathfrak{L}$. Then $S=G-T+S T \in \mathfrak{L}$. We have $(I-T)(I-S) \in \Phi^{0}(Y)$ and $S, T \in \mathfrak{L}$, so Lemma 5 (2) applies. Thus, $(I-\bar{T})(I-\bar{S}) \in \Phi^{0}(X)$. Similarly, $(I-\bar{S})(I-\bar{T}) \in \Phi^{0}(X)$. This proves that $I-\bar{T} \in \Phi(X)$ and $\operatorname{ind}(I-\bar{T})=-\operatorname{ind}(I-\bar{S})$. By Lemma 6

$$
\operatorname{ind}(I-\bar{T}) \leq \operatorname{ind}(I-T)=-\operatorname{ind}(I-S) \leq-\operatorname{ind}(I-\bar{S})=\operatorname{ind}(I-\bar{T})
$$

Lemma 8. Assume (D) holds. Suppose $T \in \mathfrak{L}, \lambda \in \mathbf{C}, \lambda \neq 0$. If $\lambda-T \in \Phi(Y)$, then $\lambda-\bar{T} \in \Phi(X)$ and $\operatorname{ind}(\lambda-T)=\operatorname{ind}(\lambda-\bar{T})$.
Proof. We may assume $\lambda=1$. By hypothesis $\exists S \in \mathscr{B}(Y)$ and $\exists F, G \in \mathscr{F}(Y)$ such that

$$
(I-S)(I-T)=I-F \quad \text { and } \quad(I-T)(I-S)=I-G
$$

Let $M=\mathscr{N}(F) \cap \mathscr{N}(G)$, and note that $M$ has finite codimension in $Y$. Since $S=-T+S T$ on $M$ it follows from (\#) that

$$
\exists J>0 \quad \text { with } \quad\|S y\|_{Y} \leq J\|y\|_{X} \quad(y \in M)
$$

Let $\bar{M}$ be the closure of $M$ in $X$ and let $\hat{S}: \bar{M} \rightarrow Y$ be the extension of $S$ acting on $M$. Now $\bar{M}$ has finite codimension in $X$, and $\hat{S}+\bar{T}-\bar{T} \hat{S}=0$ on $\bar{M}$. Write $X=\bar{M} \oplus N$ where $N$ is finite dimensional. Define $\bar{R} \in B(X)$ by $\bar{R}=\hat{S}$ on $\bar{M}$ and $\bar{R}=0$ on $N$. Note that $\bar{R}(X) \subset Y$. Since $\bar{R}+\bar{T}-\bar{T} \bar{R}$ has finite-dimensional range on $X, I-\bar{T}$ has right Fredholm inverse $I-\bar{R}$ on $X$.

Let $R$ be the restriction of $\bar{R}$ to $Y$. Then $R+T-T R \in \mathscr{F}(Y)$. A straightforward computation shows $R-S \in \mathscr{F}(Y)$. It follows that $R+T-R T \in$ $\mathscr{F}(Y)$. Since $R$ and $T$ have extensions to operators in $\mathscr{B}(X)$, it follows that $\bar{R}+\bar{T}-\bar{R} \bar{T} \in \mathscr{F}(X)$. Thus, $I-\bar{T} \in \Phi(X)$.

Concerning index, Lemma 6 implies that $\operatorname{ind}(I-\bar{T}) \leq \operatorname{ind}(I-T)$. The reverse inequality follows from [2, Theorem 4.8].

The completion of the proof of Theorem 4. First note that in order to prove part (1) of the theorem, it suffices to prove it in the two cases, when (C) holds and when (D) holds. The combination of Lemmas 5, 6, 7, and 8 establishes part (1) in these two cases. Now we prove part (2), so we assume that (D) holds. Suppose $Y=M \oplus N$ where $M$ is a subspace of $Y$ which is closed in $X$ and $N$ is a finite-dimensional subspace of $Y$. But then $M \oplus N$ is a closed subspace of $X$, a contradiction.

Assume $T \in \mathfrak{L}$. If $\bar{T} \in \Phi(X)$, then $\mathscr{R}(\bar{T}) \subseteq Y$ and has finite codimension in $Y$. It follows that $Y=\mathscr{R}(\bar{T}) \oplus N$ where $N$ is finite dimensional. This is a contradiction by the argument above. This proves $\bar{T} \notin \Phi(X)$, so

$$
0 \in \omega(\bar{T}) \subseteq W(\bar{T}) \subseteq \sigma(\bar{T}) .
$$

Now suppose $T \in \Phi(Y)$. Then $\exists S \in B(Y)$ and $\exists F \in \mathscr{F}(Y)$ with $I-F=$ $S T \in \mathfrak{L}$. Let $M=\mathscr{N}(F)$. By (\#) it follows that $M$ is closed in $X$. But $M$ has finite codimension in $Y$, so again we have a contradiction. Thus

$$
0 \in \omega(T) \subseteq W(T) \subseteq \sigma(T) .
$$

Now consider the case where $Y$ is closed in $X$, but $Y$ has no closed complement in $X$. Certainly $\bar{T} \notin \Phi(X)$, since otherwise, the assumption $\mathscr{R}(\bar{T}) \subseteq Y$ implies that $Y$ has finite codimension in $X$, a contradiction. Suppose $T \in \Phi(Y)$. Then $\exists S \in \mathscr{B}(Y)$ and $F \in \mathscr{F}(Y)$ such that $S T-F$ is the identity on $Y$. Set $\bar{P}=S \bar{T}-\bar{F}$. Then $\bar{P}$ is a bounded projection of $X$ onto $Y$. Again, this is a contradiction. Thus, $0 \in \omega(\bar{T})$ and $0 \in \omega(T)$, so equality holds between the sets as indicated in (3).

Concerning Theorem 4, note that it is always true that $0 \in \sigma(\bar{T})$, since $\mathscr{R}(\bar{T}) \subseteq Y \neq X$. However, $\sigma(T)$ need not contain 0 . For example, assume that $Y$ and $Z$ are proper closed subspaces of $X$ with $X=Y \oplus Z$. Let $\bar{P}$ be the projection of $X$ on $Y$ which is zero on $Z$. Then $\sigma(\bar{P})=\{0,1\}$. But the operator $P=\bar{P} \mid Y$ is the identity operator on $Y$, so $\sigma(P)=\{1\}$. Also, assuming that neither $Y$ nor $Z$ is finite dimensional, we have $\omega(\bar{P})=W(\bar{P})=$ $\{0,1\}$, and $\omega(P)=W(P)=\{1\}$.

We give one example where Theorem 4 applies (other applications are given in the next section).

Example 9. Let $\Omega$ be a locally compact Hausdorff space which is $\sigma$-compact, and let $\mu$ be a positive Borel measure on $\Omega$. Assume that $K(x, t)$ is a kernel defined on $\Omega \times \Omega$ with the property
( $\dagger$ ) $\quad x \rightarrow K(x, t)$ is a continuous and bounded map of $\Omega$ into $L^{1}(\Omega, \mu)$.
When $K(x, t)$ is the kernel of a locally continuous and locally compact operator $T$ on $C(\Omega)$, then $K$ satisfies ( $\dagger$ ); this is the key condition 12.7(a) in
[10, Theorem 12.2, p. 303]. It is easy to see that

$$
T_{\infty}(f)(x)=\int_{\Omega} K(x, t) f(t) d \mu(t) \quad\left(f \in L^{\infty}\right)
$$

defines an operator in $\mathscr{B}\left(L^{\infty}(\Omega, \mu)\right)$ with the property that $T_{\infty}\left(L^{\infty}\right) \subseteq C(\Omega)$. Then Theorem 4 implies that the spectral and Fredholm properties of $T$ in $C(\Omega)$ and the spectral theory of $T_{\infty}$ in $L^{\infty}$ are the same. This is essentially the content of [10, Theorem 12.7, p. 321].

## 3. Applications to operators on Lebesgue spaces

Let $(\Omega, \mu)$ be a measure space with $\mu$ a positive $\sigma$-finite measure on $\Omega$. Assume $T$ is a linear map defined on the space of integrable simple functions on $\Omega$ with values in the measurable functions on $\Omega$, and suppose $T$ has a bounded extension $T_{r}$ on $L^{r}(\boldsymbol{\Omega})$ for all $r$ in the interval $[p, s]$ (here $1 \leq p<s<\infty$ ). When $\mu(\Omega)$ is finite, then it is well known that

$$
\begin{equation*}
\sigma\left(T_{r}\right) \subseteq \sigma\left(T_{p}\right) \cup \sigma\left(T_{s}\right) \quad \text { for } r \in[p, s] \tag{*}
\end{equation*}
$$

In the general case, only the less precise inclusion holds that

$$
\sigma\left(T_{r}\right) \subseteq\left[\sigma\left(T_{p}\right) \cup \sigma\left(T_{s}\right)\right]^{\wedge} \quad \text { for } r \in[p, s]
$$

where $\hat{E}$ denotes the polynomial convex hull of the set $E$; see [2, Theorem 5.3]. For example, when $\Omega=[0, \infty)$ and $\mu$ is Lebesgue measure, let $T$ be defined by the formula

$$
T(f)(x)=x^{-1} \int_{0}^{x} f(t) d t, \quad x>0
$$

Then $T$ has a bounded extension $T_{p}$ on $L^{p}([0, \infty))$ for $1<p \leq \infty$, and it is known that when $1<p<\infty$, then $\sigma\left(T_{p}\right)$ is the circle with center and radius $\left(2\left(1-p^{-1}\right)\right)^{-1}$ [4]. The inclusion indicated in (*) does not hold in this example.

Now we apply the results of $\S 2$ to prove that in some cases (*) can be preserved. Let $L^{r, \infty}$ be the space $L^{r, \infty}=L^{r}(\Omega, \mu) \cap L^{\infty}(\Omega, \mu)$ with the complete norm $\max \left(\|f\|_{r},\|f\|_{\infty}\right)$. Fix $1 \leq p<s<\infty$. For $r, t \in[p, s]$ with $r \leq t$ we have $L^{r, \infty} \subseteq L^{t, \infty}$. Assume that $T$ is a linear map with extensions $T_{r, \infty} \in \mathscr{B}\left(L^{r, \infty}\right)$ for $r \in[p, s]$. The key fact used to establish (*) when $\Omega$ has finite measure is that in this case when $1 \leq r \leq t \leq \infty$, then $L^{t} \subseteq L^{r}$. Using the fact that there is a similar ordering of the spaces $L^{r, \infty}$, the same analysis as in the finite measure case [2, §4] proves that when $p \leq r \leq s$ :
(i) $\sigma\left(T_{r, \infty}\right) \subseteq \sigma\left(T_{p, \infty}\right) \cup \sigma\left(T_{s, \infty}\right)$;
(ii) $W\left(T_{r, \infty}\right) \subseteq W\left(T_{p, \infty}\right) \cup W\left(T_{s, \infty}\right)$;
(iii) $\omega\left(T_{r, \infty}\right) \subseteq \omega\left(T_{p, \infty}\right) \cup \omega\left(T_{s, \infty}\right) \cup \omega_{0}$ where $\omega_{0}=\left\{\lambda \notin \omega\left(T_{p, \infty}\right) \cup\right.$ $\left.\omega\left(T_{s, \infty}\right): \operatorname{ind}\left(\lambda-T_{p, \infty}\right) \neq \operatorname{ind}\left(\lambda-T_{s, \infty}\right)\right\}$.

Theorem 10. Assume a linear operator $T$ has bounded extensions $T_{p} \in \mathscr{B}\left(L^{p}\right)$ and $T_{s} \in \mathscr{B}\left(L^{s}\right)$ where $1 \leq p<s<\infty$. In addition assume $T_{p}\left(L^{p}\right) \subseteq L^{\infty}$ and $T_{s}\left(L^{s}\right) \subseteq L^{\infty}$. Then for $p \leq r \leq s:$
(1) $\sigma\left(T_{r}\right) \subseteq \sigma\left(T_{p}\right) \cup \sigma\left(T_{s}\right)$;
(2) $W\left(T_{r}\right) \subseteq W\left(T_{p}\right) \cup W\left(T_{s}\right)$;
(3) $\omega\left(T_{r}\right) \subseteq \omega\left(T_{p}\right) \cup \omega\left(T_{s}\right) \cup \omega_{0}$ where $\omega_{0}=\left\{\lambda \notin \omega\left(T_{p}\right) \cup \omega\left(T_{s}\right)\right.$ : $\left.\operatorname{ind}\left(\lambda-T_{p}\right) \neq \operatorname{ind}\left(\lambda-T_{s}\right)\right\}$.

Furthermore, if $p \leq r \leq t \leq s$ and $\lambda-T_{k} \in \Phi\left(L^{k}\right)$ for $k=r$ and $k=t$, then $\operatorname{ind}\left(\lambda-T_{r}\right) \leq \operatorname{ind}\left(\lambda-T_{t}\right)$.

Proof. By the Riesz Convexity Theorem [6, Theorem 11, p. 525] it follows that when $r \in[p, s]$, then $T_{r} \in \mathscr{B}\left(L^{r}\right)$, and also that $T_{r}\left(L^{r}\right) \subseteq L^{\infty}$. The restriction operators $T_{r, \infty}$ satisfy properties (i), (ii), and (iii) listed just prior to the theorem. By applying Theorem 4, we have that the spectral and Fredholm properties of $T_{r}$ and $T_{r, \infty}$ are exactly the same. This proves (1)-(3).

Assume $p \leq r \leq t \leq s$ and $\lambda-T_{k} \in \Phi\left(L^{k}\right)$ for $k=r$ and $k=t$. By Theorem 4, $\lambda-T_{k, \infty} \in \Phi\left(L^{k, \infty}\right)$ for $k=r$ and $k=t$. Then [2, Theorem 4.8] implies that $\operatorname{ind}\left(\lambda-T_{r, \infty}\right) \leq \operatorname{ind}\left(\lambda-T_{t, \infty}\right)$. Therefore Theorem 4 gives $\operatorname{ind}\left(\lambda-T_{r}\right) \leq \operatorname{ind}\left(\lambda-T_{t}\right)$.

The situation described in the hypotheses of Theorem 10 is quite common. We give one example. Let $G$ be unimodular locally compact group with a fixed Haar measure. We write the group operation on $G$ as multiplication. Let $k \in L^{1}(G) \cap L^{2}(G)$, and assume $\varphi$ and $\psi$ are in $L^{\infty}(G)$. Let $T$ be the integral operator determined by the kernel $\varphi(x) k\left(x t^{-1}\right) \psi(t)$. Using [8, (20.19) (iii), (iv)], it follows that $T_{2}\left(L^{2}\right) \subseteq L^{\infty}$ and $T_{\infty}\left(L^{\infty}\right) \subseteq L^{\infty}$, so $T_{s}\left(L^{s}\right) \subseteq L^{\infty}$ for $2 \leq s \leq \infty$. Thus, Theorem 10 applies to $T_{r}$ with $r$ in any interval [2,s], $s<\infty$.

Other spaces can serve in the role played by $L^{\infty}$ above. For example, assume $\psi(t) \in L^{\infty}([0, \infty))$ and $x \varphi(x) \in L^{\infty} \cap L^{2}([0, \infty))$. Define $T$ by

$$
T(f)(x)=\varphi(x) \int_{0}^{x} \psi(t) f(t) d t=(x \varphi(x)) x^{-1} \int_{0}^{x} \psi(t) f(t) d t, \quad x>0
$$

Then $T_{2} \in \mathscr{B}\left(L^{2}\right), T_{\infty}\left(L^{\infty}\right) \subseteq L^{2}$. Therefore for $2 \leq r \leq \infty, T_{r}\left(L^{r}\right) \subseteq L^{2}$. The spaces $L^{2, r}=L^{2} \cap L^{r}$ are ordered by $L^{2, t} \subseteq L^{2, r}$ when $2 \leq r \leq t \leq \infty$.The same type of analysis used above applied to the operators $T_{2, r} \in \mathscr{B}\left(L^{2, r}\right)$ and their extensions $T_{r} \in \mathscr{B}\left(L^{r}\right)$, leads to a result similar to Theorem 10. In particular, when $2 \leq p \leq r \leq s<\infty$,

$$
\sigma\left(T_{r}\right) \subseteq \sigma\left(T_{p}\right) \cup \sigma\left(T_{s}\right)
$$

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