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# A REMARK ON RADON-NIKODYM PROPERTIES OF ORDERED HILBERT SPACES

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ABSTRACT. We characterize classical $L^2$-spaces as well as $L^2$-spaces of $W^*$-algebras by appropriate Radon- Nikodym principles.

Segal [9] has shown that usual measure theory can be extended reasonably only to measure spaces, whose corresponding $L^1$-spaces have the Radon–Nikodym property. The aim of this note is to show that $L^2$-spaces obtained via commutative, respectively, noncommutative integration are as well determined as ordered spaces by suitable Radon–Nikodym principles. Order theoretical characterizations of classical $L^2$-spaces were obtained by Sz.-Nagy [11] in the separable and Schaefer [5] in the general case. Noncommutative $L^2$-spaces were characterized by Connes [2] and Wittstock and the author [8, 6]. Radon–Nikodym properties of these $L^2$-spaces were established by Connes [2], Stratila–Zsido [10] and in [7].

Suppose $\mathscr{H}$ is a Hilbert space, which is ordered by a self-dual cone $\mathscr{H}^+$. Let $J$ be the antilinear unitary involution associated with $\mathscr{H}^+$ by [2, Proposition 4.1]. Furthermore let $\mathscr{M}_h$ be the ideal center of $(\mathscr{H}, \mathscr{H}^+)$ in the sense of Wils—see [1, p. 76]—and $\mathscr{M} = \mathscr{M}_h \oplus i\mathscr{M}_h$. Bös [1] has shown that $\mathscr{M}$ is a commutative $W^*$-algebra with involution $x^* = JxJ$, $x \in \mathscr{M}$. In particular the cones of operator-positive, respectively, order-positive elements in $\mathscr{M}$ coincide.

**Definition.** $(\mathscr{H}, \mathscr{H}^+)$ has the Radon–Nikodym property if for $\eta$, $\xi \in \mathscr{H}^+$ satisfying $\eta \leq \xi$ there exists $x \in \mathscr{M}^+$ such that $\eta = x\xi$.

**Proposition 1.** *$(\mathscr{H}, \mathscr{H}^+)$ has the Radon–Nikodym property if and only if $(\mathscr{H}, \mathscr{H}^+)$ is unitarily equivalent to $(L^2(X, \mu), L^2(X, \mu)^+)$ for a measure space $(X, \mu)$ in the sense of Segal* [9].

We shall give the proof of Proposition 1 in a moment. If $V$ is a linear space, then we shall write $V_n = M_n(V)$ for the space of $n \times n$ matrices with entries in $V$. Again let $\mathscr{H}$ be a Hilbert space and suppose that $\{\mathscr{H}_n^+, n \in \mathbf{N}\}$ is a

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family of self-dual cones $\mathscr{H}_n^+ \subseteq \mathscr{H}_n$ such that $(\mathscr{H}, \mathscr{H}_n^+)$ is a matrix ordered space. Let $J_n$ be the involution in $\mathscr{H}_n$ associated with $\mathscr{H}_n^+$. Finally let $\mathscr{M}$ be the matrix-multiplier algebra of $(\mathscr{H}, \mathscr{H}_n^+, n \in \mathbf{N})$ in the sense of [8, Definition 2.1].

**Definition.** $(\mathscr{H}, \mathscr{H}_n^+, n \in \mathbf{N})$ has the matricial Radon–Nikodym property if for $n \in \mathbf{N}, \xi, \eta \in \mathscr{H}_n^+$ satisfying $\eta \leq \xi$ there exists $x \in \mathscr{M}_n^+$ such that $\eta = xJ_nxJ_n\xi$.

**Proposition 2.** *$(\mathscr{H}, \mathscr{H}_n^+, n \in \mathbf{N})$ has the matricial Radon–Nikodym property if and only if $(\mathscr{M}, \mathscr{H}, \mathscr{H}_n^+, n \in \mathbf{N})$ is a matrix-ordered standard form in the sense of* [8, Definition 1.4].

*Proof of Proposition* 1. Suppose $(\mathscr{H}, \mathscr{H}^+)$ has the Radon–Nikodym property. Let $\xi_\alpha, \eta_\alpha \in \mathscr{H}^+$; $\alpha, \beta = 1, 2$ and $\xi = \xi_1 + \xi_2 = \eta_1 + \eta_2$. Then $\xi_\alpha = x_\alpha \xi$, $\eta_\beta = y_\beta \xi$ for $x_\alpha, y_\alpha \in \mathscr{M}^+$. Let $\rho_{\alpha\beta} = x_\alpha y_\beta \xi$. Then $\xi_\alpha = \sum_\beta \rho_{\alpha\beta}$ and $\eta_\beta = \sum_\alpha \rho_{\alpha\beta}$. Hence $(\mathscr{H}, \mathscr{H}^+)$ has the Riesz decomposition property and therefore satisfies the hypothesis of [6, Corollary 2.6]. There is one and only one way to define a matrix order on $\mathscr{H}$ with by uniqueness self-dual cones $\mathscr{H}_n^+ \subseteq \mathscr{H}_n$ such that $\mathscr{H}_1^+ = \mathscr{H}^+$. $(\mathscr{H}, \mathscr{H}_n^+, n \in \mathbf{N})$ satisfies the hypothesis (IV) of [6, Theorem 3.2]. Let $\mathscr{N}$ be the matrix multiplier algebra of $(\mathscr{H}, \mathscr{H}_n^+)$. Since the cones $\mathscr{H}_n^+$ coincide with their transpose

$$\mathscr{H}_n^{+\mathrm{T}} = \{(\xi_{ij}) | (\xi_{ji}) \in \mathscr{H}_n^+\}$$

and $(\mathscr{N}', \mathscr{H}, \mathscr{H}_n^{+\mathrm{T}})$ is a matrix ordered standard form we have $\mathscr{N} = \mathscr{N}'$. Hence $\mathscr{N} = \mathscr{M}$ is a commutative $W^*$-algebra in standard form. Applying [3, Theorem 2.3] and [4, Proposition 1.18.1] completes the proof.

Note that the above argument can also be used to show that a Hilbert lattice is order isomorphic to an $L^2$-space.

*Proof of Proposition* 2. Suppose $(\mathscr{H}, \mathscr{H}_n^+, n \in \mathbf{N})$ has the matricial Radon–Nikodym property. Let $n \in \mathbf{N}$ be fixed, $\xi, \eta \in \mathbf{H}_n^+$ such that $\xi \perp \eta$. Then $\xi = xJ_nxJ_n(\xi + \eta)$ for some $x \in \mathscr{M}_n^+$. Now we have

$$\left\langle \begin{pmatrix} xJ_nxJ_n\eta & x\eta \\ J_nxJ_n\eta & \eta \end{pmatrix}, \quad \eta \otimes \gamma \right\rangle \geq 0 \quad \text{for } \gamma \in \mathscr{M}_2(\mathbf{C})^+.$$

Hence

$$\begin{pmatrix} 0 & \langle x\eta, \eta\rangle \\ \langle J_nxJ_n\eta\rangle & \langle \eta, \eta\rangle \end{pmatrix}$$

is a positive matrix and $\langle x\eta, \eta\rangle = 0 = \langle p\eta, \eta\rangle$ where $p = \operatorname{supp}(x) \in \mathscr{M}$ by [8, Theorem 2.2]. Hence we have $p\eta = J_npJ_n\eta = 0$ and $\xi = p\xi = J_npJ_n\xi$. Now one can prove [8, Lemma 4.1; Theorem 4.3 (ii) $\Rightarrow$ (i)] under the above hypothesis. The converse direction in the proof of Proposition 2 is established in [7].

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