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STABILITY OF SURFACES WITH CONSTANT MEAN CURVATURE

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ABSTRACT. We estimate the Gaussian curvature of a conformal metric on a surface of constant mean curvature in space form $M^3(c)$. By use of the estimates, we study stability of surfaces with constant mean curvature in $M^3(c)$.

1. INTRODUCTION

Let $M^3(c)$ be the three-dimensional space form of constant sectional curvature c. Let M be a surface with constant mean curvature H in $M^3(c)$, g be the induced metric, and K be the Gaussian curvature of g. We get the following results:

Theorem 1. *The Gaussian curvature* $\overline{K}$ *of the conformal metric* $\bar{g} = \sigma g$ *satisfies* $\overline{K} \leq 1$, *where*

$$\sigma = \begin{cases} 2H^2 - K + 2c, \ \textit{when } c \geq 0 \\ -K, \ \textit{when } c < 0 \textit{ and } H^2 + c \leq 0 \end{cases} \tag{1.1}$$

and $\overline{K} \equiv 1$ *if and only if* $c = 0$ *and* $H = 0$, *or* $c < 0$ *and* $H^2 + c = 0$.

Corollary 1.1 (Proposition 2.2 of [1]). *Let* M *be a minimal surface of* $M^3(c)$. *Then the Gaussian curvature* $\overline{K}$ *of the conformal metric* $\bar{g} = \sigma g$ *satisfies* $\overline{K} \leq 1$, *where* $\sigma = 2c - K$, *when* $c > 0$, *and* $\sigma = -K$, *when* $c \leq 0$.

Let $X : M \to M^3(c)$ be an immersion with constant mean curvature H. Let $D \subset M$ be a domain in M with compact closure $\overline{D}$ and piecewise smooth boundary ∂D. Following §5 of [4], we say that D is *strongly stable* if

$$I(f) = \int_D [|\nabla f|^2 - 2(2c + 2H^2 - K)f^2] \, dA > 0 \tag{1.2}$$

for all functions $f : D \to R$ such that $f|_{\partial D} = 0$.

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Making use of Theorem 1, we obtain:

Theroem 2. *Let $X : M \to M^3(c)$ $(c \geq 0)$ be an immersion with constant mean curvature H. Assume that $D \subset M$ is simply connected and that*

$$\int_{\overline{D}} (2c - K + 2H^2)\, dA < 2\pi . \tag{1.3}$$

Then D is strongly stable.

Theorem 3. *Let $X : M \to H^3(c)$ $(c < 0)$ be an immersion with constant mean curvature H and $H^2 + c \leq 0$. Assume that $D \subset M$ is simply connected and that*

$$\int_{\overline{D}} -K\, dA < 2\pi . \tag{1.4}$$

Then D is strongly stable.

From definition of strongly stable, when $H = 0$, we easily see that *strongly stable* reduces to *stable* of minimal surfaces. We get from Theorem 2 and Theorem 3:

Corollary 2.1 ([5], Theorem 1.2 of [1]). *Let $X : M \to M^3(c)$ $(c \geq 0)$ be a minimal immersion. Assume that $D \subset M$ is simply connected and $\int_{\overline{D}} (2c - K)\, dA < 2\pi$. Then D is stable.*

Corollary 3.1 (Theorem 1.3 of [1]). *Let $X : M \to H^3(c)$ $(c < 0)$ be a minimal immersion. Assume that $D \subset M$ is simply connected and $\int_{\overline{D}} |K|\, dA < 2\pi$. Then D is stable.*

Corollary 3.2 (Proposition 5.2 of [4]). *Let $X : M \to H^3(-1)$ be an immersion with constant mean curvature one. Let $D \subset M$ be a simply connected compact domain. If $\int_D -K\, dA < 2\pi$, then D is strongly stable.*

Let M is a minimal surface; it is a well known that the Gaussian curvature $\overline{K} \equiv 1$ of $\bar{g} = -Kg$. We now generalize the result to surfaces with constant mean curvature in $M^3(c)$.

Theorem 4. *Let M be a surface with constant mean curvature H in $M^3(c)$ and M is not totally umbilic. Then the Gaussian curvature $\overline{K}$ of $\bar{g} = \sigma g$ satisfies*

$$\overline{K} = 1 - \frac{H^2 + c}{H^2 - K + c} \tag{1.5}$$

where $\sigma = H^2 - K + c > 0$, and $\overline{K} \equiv 1$ if and only if $H^2 + c = 0$.

Corollary 4.1. *Let M be a minimal surface in R^3. Then the Gaussian curvature $\overline{K} \equiv 1$ of $\bar{g} = -Kg$.*

Corollary 4.2 (Proposition 3 of [7]). *Let M be a surface with constant mean curvature one in $H^3(-1)$. Then the Gaussian curvature $\overline{K} \equiv 1$ of $\bar{g} = -Kg$.*

2. Fundamental formulas

Let M be a surface in $M^3(c)$ and let e_1, e_2, e_3 be a local field of orthonormal frames in $M^3(c)$, such that, restricted to M, the vector field e_3 is normal to M. Then, the second fundamental form B and the mean curvature H for M can be written as

$$B = \sum_{i,j} h_{ij}\omega_i\omega_j e_3, \qquad H = \frac{1}{2}\sum_i h_{ii}. \tag{2.1}$$

The Gauss–Codazzi equations for M are

$$K = c + 2H^2 - |B|^2/2, \qquad \text{where } |B|^2 = \sum_{i,j} h_{ij}^2 \tag{2.2}$$

$$h_{ijk} = h_{ikj} \qquad (1 \le i, j, k, \cdots \le 2). \tag{2.3}$$

We denote by Δ the Laplacian relative to the induced metric on M. If $H =$ constant, then ([2])

$$\tfrac{1}{2}\Delta|B|^2 = |\nabla B|^2 - |B|^4 + 2c|B|^2 - 4cH^2 + 2HW \tag{2.4}$$

where

$$|\nabla B|^2 = \sum_{i,j,k}(h_{ijk})^2, \qquad W = \sum_{i,j,k} h_{ij}h_{jk}h_{ki}. \tag{2.5}$$

We get by a direct computation

$$2HW = 6H^2|B|^2 - 8H^4, \tag{2.6}$$

$$\because \lambda_1 + \lambda_2 = 2H, \; \therefore \lambda_1\lambda_2 = 2H^2 - \overset{\frac{|B|^2}{2}}{\sum}$$

$$\begin{aligned} \therefore 2HW &= 2H(\lambda_1^3 + \lambda_2^3) \\ &= 2H(\lambda_1 + \lambda_2)(\lambda_1^2 + \lambda_2^2 - \lambda_1\lambda_2) \\ &= 6H^2|B|^2 - 8H^4. \end{aligned}$$

From (2.4) and (2.6), we have

$$\tfrac{1}{2}\Delta|B|^2 = |\nabla B|^2 - |B|^4 + 2c|B|^2 - 4cH^2 + 6H^2|B|^2 - 8H^4. \tag{2.7}$$

Proposition 2.1. *Let M be a surface with $H =$ constant in $M^3(c)$, then*

$$|\nabla(|B|^2)|^2 = 2(|B|^2 - 2H^2)\cdot|\nabla B|^2 \le 2|B|^2\cdot|\nabla B|^2. \tag{2.8}$$

Proof. At any point of M, let $h_{ij} = \lambda_i\delta_{ij}$, we have

$$\begin{aligned} |\nabla(|B|^2)|^2 &= 4\sum_k\left(\sum_{i,j} h_{ij}h_{ijk}\right)^2 = 4\sum_k\left(\sum_i \lambda_i h_{iik}\right)^2 \\ &= 4\sum_k(\lambda_1 h_{11k} + \lambda_2 h_{22k})^2. \end{aligned} \tag{2.9}$$

But $H =$ constant implies

$$h_{11k} + h_{22k} = 0, \qquad \lambda_1 + \lambda_2 = 2H. \tag{2.10}$$

(2.9) and (2.10) yield

$$\begin{aligned} |\nabla(|B|^2)|^2 &= 4(\lambda_1 - \lambda_2)^2 \sum_k h_{11k}^2 = 2(\lambda_1 - \lambda_2)^2 \sum_{i,k} h_{iik}^2 \\ &= 4(|B|^2 - 2H^2) \sum_{i,k} h_{iik}^2. \end{aligned} \tag{2.11}$$

On the other hand, we easily establish by a direct computation:

$$\begin{aligned} |\nabla B|^2 &= 3\sum_{i\neq k} h_{iik}^2 + \sum_k h_{kkk}^2 = 2\sum_{i\neq k} h_{iik}^2 + \sum_{i,k} h_{iik}^2 \\ &= 2\sum_{i,k} h_{iik}^2. \end{aligned} \tag{2.12}$$

Combining (2.11) with (2.12), we obtain (2.8). Q.E.D.

3. Proofs of Theorem 1, 2, 3 and 4

(3.1) *Proof of Theorem* 1. Case $c \geq 0$: (1.1) and (2.2) yield

$$\sigma = 2H^2 - K + 2c = c + |B|^2/2 > 0, \tag{3.2}$$

where $c = 0$; we assume that M is not totally geodesic. Thus we can define a conformal metric $\overline{g} = \sigma g$ on M. As well known, the Gaussian curvature $\overline{K}$ of $\overline{g}$ satisfies ([3]):

$$\sigma\overline{K} = K - \tfrac{1}{2}\Delta \log \sigma. \tag{3.3}$$

By (3.2) and (3.3), we have

$$-\sigma\overline{K} = \sigma - 2(c + H^2) + \frac{1}{2}\frac{\Delta\sigma}{\sigma} - \frac{1}{2\sigma^2}|\nabla\sigma|^2. \tag{3.4}$$

From (2.7), (3.2) and Proposition 2.1, we get

$$\begin{aligned} \tfrac{1}{2}\Delta\sigma &= \tfrac{1}{4}\Delta(|B|^2) = \tfrac{1}{2}|\nabla B|^2 - \tfrac{1}{2}|B|^4 + c|B|^2 - 2cH^2 + 3H^2|B|^2 - 4H^4 \\ &\geq \tfrac{1}{2}|\nabla\sigma|^2/\sigma - 2\sigma^2 + 6\sigma c + 6H^2\sigma - 4c^2 - 8H^2c - 4H^4 \end{aligned} \tag{3.5}$$

and equality holds if and only if $c = 0$ and $H = 0$.

Noting $\frac{1}{4}(\lambda_1 - \lambda_2)^2 = |B|^2/2 - H^2 \geq 0$, then

$$\sigma = c + |B|^2/2 \geq c + H^2. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\begin{aligned} \tfrac{1}{2}\Delta\sigma &\geq \tfrac{1}{2}|\nabla\sigma|^2/\sigma - 2\sigma^2 + 2\sigma c + 2H^2\sigma + 4(c + H^2)c + 4H^2(c + H^2) \\ &\quad - 4c^2 - 8H^2c - 4H^4 \\ &= \tfrac{1}{2}|\nabla\sigma|^2/\sigma - 2\sigma^2 + 2\sigma c + 2H^2\sigma. \end{aligned} \tag{3.7}$$

Combining (3.4) with (3.7), we get $\overline{K} \leq 1$ and $\overline{K} \equiv 1$ if and only if $c = 0$ and $H = 0$.

Case $c < 0$: Assume $H^2 + c \leq 0$ and M is not totally umbilic. Then $\frac{1}{4}(\lambda_1 - \lambda_2)^2 = |B|^2/2 - H^2 > 0$, and

$$\sigma = -K = -c - 2H^2 + |B|^2/2 > 0. \tag{3.8}$$

Thus we can define a conformal metric $\overline{g} = \sigma g$ on M.

(3.3) and (3.8) yield

$$-\sigma\overline{K} = \sigma + \frac{1}{2}\frac{\Delta\sigma}{\sigma} - \frac{1}{2\sigma^2}|\nabla\sigma|^2. \tag{3.9}$$

From (2.7), (3.8), and Proposition 2.1, we get

$$\begin{aligned}\frac{1}{2}\Delta\sigma &= \frac{1}{4}\Delta(|B|^2) \\ &= \frac{1}{2}2\frac{|\nabla(|B|^2)|^2}{(|B|^2 - 2H^2)} - \frac{1}{2}|B|^4 + c|B|^2 - 2cH^2 + 3H^2|B|^2 - 4H^4 \\ &\geq \frac{1}{2}\frac{|\nabla\sigma|^2}{\sigma} - 2\sigma^2 - 2\sigma(H^2 + c) \geq \frac{1}{2}\frac{|\nabla\sigma|^2}{\sigma} - 2\sigma^2.\end{aligned} \tag{3.10}$$

Combining (3.9) with (3.10), we have $\overline{K} \leq 1$, and $\overline{K} \equiv 1$ if and only if $H^2 + c = 0$. Q.E.D.

(3.11) *Proof of Theorem* 2. Assume that D is not *strongly stable*. By the Smale's version of the Morse index theorem [6], there exists a domain $D' \subset D$ and a function $f : D' \to (0, \infty)$ so that $\Delta f - 2f(K - 2c - 2H^2) = 0$ in D', and $f|_{\partial D'} = 0$. Let g be the induced metric, from Theorem 1, the Gaussian curvature $\overline{K} \leq 1$ of $\overline{g} = (2H^2 + 2c - K)g$. By Proposition 3.13 of [1], $2 \geq \lambda_1(D^*)$, where D^* is a geodesic disk in a sphere $S^2(1)$ with curvature 1 and area of D^* is equal to the area of D' in the metric $\overline{g}$. Here $\lambda_1(D^*)$ is the first eigenvalue of the Laplacian of the sphere $S^2(1)$ on D^*. Since $\int_{\overline{D}}(2H^2 + 2c - K)\, dA < 2\pi$, the area of D' in the metric $\overline{g}$ is smaller than 2π. It follows that D^* is contained in a hemisphere of $S^2(1)$ the first eigenvalue of which is 2. Thus

$$2 \geq \lambda_1(D^*) > 2,$$

which is a contradiction. Q.E.D.

(3.12) *Proof of Theorem* 3. We first observe that, since $c < 0$ and $H^2 + c \leq 0$,

$$\begin{aligned}I_D(f) &= \int_{\overline{D}}[|\nabla f|^2 - 2(2c + 2H^2 - K)f^2]\, dA \\ &\geq \int_D [|\nabla f|^2 + 2Kf^2]\, dA = \tilde{I}_D(f)\end{aligned} \tag{3.13}$$

for all functions $f : D \to R$ such that $f|_{\partial D} = 0$.

To show that D is *strongly stable*, it suffices to show $\widetilde{I}_D(f) > 0$ for all such f. The proof is similar to the proofs of Theorem 1.3 of [1] and above Theroem 2. We omit it here. Q.E.D.

(3.14) *Proof of Theorem* 4. We assume M is not totally umbilic, then

$$\sigma = H^2 - K + c = |B|^2/2 - H^2 = \tfrac{1}{4}(\lambda_1 - \lambda_2)^2 > 0. \tag{3.15}$$

We can define a conformal metric $\overline{g} = \sigma g$ on M.

(3.3) and (3.15) yield:

$$-\sigma\overline{K} = \sigma - H^2 - c + \frac{1}{2}\frac{\Delta\sigma}{\sigma} - \frac{1}{2\sigma^2}|\nabla\sigma|^2. \tag{3.16}$$

From (2.7), (3.15) and Proposition 2.1, we get by a direct computation:

$$\frac{1}{2}\Delta\sigma = \frac{1}{2}\frac{|\nabla\sigma|^2}{\sigma} - 2\sigma^2 + 2\sigma(H^2 + c). \tag{3.17}$$

Combining (3.16) with (3.17), we have

$$-\sigma\overline{K} = -\sigma + H^2 + c, \text{ i.e. } \overline{K} = 1 - \frac{H^2 + c}{H^2 - K + c} \tag{3.18}$$

and $\overline{K} \equiv 1$ if and only if $H^2 + c = 0$. Q.E.D.

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