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ULTRAPRODUCTS, ε-MULTIPLIERS, AND ISOMORPHISMS

MICHAEL CAMBERN AND KRZYSZTOF JAROSZ

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ABSTRACT. For a compact Hausdorff space X and Banach dual E^*, denote by $C(X,(E^*,\sigma^*))$ the Banach space of all continuous functions on X to E^* when the latter space is provided with its weak* topology. We show that if E_i^*, $i = 1,2$, belong to a class of Banach duals satisfying a condition involving the space of multipliers on E_i^*, then the existence of an isomorphism T mapping $C(X_1,(E_1^*,\sigma^*))$ onto $C(X_2,(E_2^*,\sigma^*))$ with $\|T\|\,\|T^{-1}\|$ small implies that X_1 and X_2 are homeomorphic. Ultraproducts of Banach spaces and the notion of ε-multipliers play key roles in obtaining this result.

1. INTRODUCTION

It has long been known that the conclusion of the classical Banach–Stone theorem regarding the topological invariance of the compact Hausdorff space X under isometries of the space $C(X)$ remains valid when isometries are replaced by small-bound isomorphisms [1, 9, 10]. Isometric Banach-Stone theorems for the space $C(X,E)$, consisting of norm-continuous vector functions on X to a Banach space E, were initiated by Jerison [24] and studied by many authors. These results were compiled in the book by Behrends [4], and much more recently have found a formulation valid for isomorphisms [7, 22, 23]. In this article we consider spaces of weak* continuous vector functions. Theorems concerning isometries of such spaces were obtained in [15]. Here we show that an isomorphic result is also possible.

If E^* is a Banach dual we denote by $C(X,(E^*,\sigma^*))$ the space of all continuous functions F on X to E^* when the latter space is provided with its weak* topology, normed by $\|F\|_\infty = \sup_{x\in X}\|F(x)\|$. This space arises quite naturally within a variety of mathematical contexts. In [12] it is shown that the characterization of the bidual of $C(X)$ originally obtained by Kakutani [25], and studied by Arens [2] and Kaplan [26], can be formulated for spaces of norm-continuous vector functions via the introduction of $C(X,(E^*,\sigma^*))$. The dual of the Bochner space $L^1(\mu,E)$ is always of the form $C(X,(E^*,\sigma^*))$ [13, Remark] (whereas $L^\infty(\mu,E^*)$ fulfills this role only with an assumption regarding the Radon–Nikodyn property [16, p. 98]). $C(X,(E^*,\sigma^*))$ provides

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the dual of a space of vector measures [13] in a manner which parallels the duality obtained for spaces of scalar measures by Gordon [19]. And the results of Dixmier and Grothendieck [17, 20] characterizing those spaces $C(X)$ which are Banach duals have vector analogues which involve $C(X, (E^*, \sigma^*))$ [14].

We will show that given compact Hausdorff spaces X_1, X_2 and Banach duals E_1^*, E_2^* which satisfy a geometric condition, then the existence of an isomorphism S mapping $C(X_1, (E_1^*, \sigma^*))$ onto $C(X_2, (E_2^*, \sigma^*))$ with $\|S\| \|S^{-1}\|$ small implies that X_1 and X_2 are homeomorphic. The only result of this nature known to the authors is found in [11], where it is assumed that the X_i are extremally disconnected and the E_i^* uniformly convex. Here we remove the assumption concerning the extremally disconnected nature of the X_i, and the geometric condition we impose is much less restrictive than the requirement of uniform convexity.

Our results depend heavily upon the concept of a multiplier on a Banach space E. (For the definition and properties of multipliers we refer to [4].) The space of multipliers on E is denoted by $\mathrm{Mult}(E)$, while $\mathscr{B}(E)$ stands for the space of all bounded operators on E. Here we employ the notion of ε-multipliers which, for each $\varepsilon > 0$, constitute a subset $\mathrm{Mult}_\varepsilon(E)$ of $\mathscr{B}(E)$ containing the unit ball in $\mathrm{Mult}(E)$. (Our use of the notation $\mathrm{Mult}_\varepsilon(E)$ can be seen to agree with that of [7].) The geometric condition which will be imposed on dual spaces is essentially that, as ε tends to 0, $\mathrm{Mult}_\varepsilon(E)$ comes ever closer to a trivial set of multipliers consisting of scalar multiples of the identity operator. In this case the unit ball of $\mathrm{Mult}(E)$, which is the intersection over all $\varepsilon > 0$ of $\mathrm{Mult}_\varepsilon(E)$, consists only of scalar multiples of the identity, and $\mathrm{Mult}(E)$ will be called *geneologically trivial.*

Our arguments are also much dependent upon the notion of an ultraproduct of Banach spaces. Here we follow the notation and terminology of [21], except that for us any ultrafilter $\mathscr{F}$ considered is invariably a free ultrafilter on the set $\mathbb{N}$ of natural numbers. Thus the ultraproduct $(E_n)_{\mathscr{F}}$ of a family of Banach spaces $(E_n)_{n\in\mathbb{N}}$ is the quotient space $l^\infty(\mathbb{N}, E_n)/N_{\mathscr{F}}$, where $N_{\mathscr{F}}$ is the subspace consisting of those elements $(e_n) \in l^\infty(\mathbb{N}, E_n)$ with $\lim_{\mathscr{F}} \|e_n\| = 0$. Here $(e_n)_{\mathscr{F}}$ denotes the equivalence class of (e_n) in $(E_n)_{\mathscr{F}}$ and $\|(e_n)_{\mathscr{F}}\| = \lim_{\mathscr{F}} \|e_n\|$, [21, p. 75]. If all E_n are equal to some fixed Banach space E, the ultraproduct is called an ultrapower, denoted by $(E)_{\mathscr{F}}$. And given operators $T_n \in \mathscr{B}(E_n)$ with $\sup_n \|T_n\| < \infty$, the operator on $(E_n)_{\mathscr{F}}$ defined by $(e_n)_{\mathscr{F}} \to (T_n e_n)_{\mathscr{F}}$ is called the ultraproduct of the family $(T_n)_{n\in\mathbb{N}}$ and is denoted by $(T_n)_{\mathscr{F}}$. Moreover $\|(T_n)_{\mathscr{F}}\| = \lim_{\mathscr{F}} \|T_n\|$.

Finally, throughout the article, if we are given any Banach space E the associated scalar field will be denoted by $\mathbb{K}$. Thus $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

2. ε-MULTIPLIERS

Definition 1. Given $\varepsilon > 0$ and $T \in \mathscr{B}(E)$ we call T an ε-multiplier if for any $e_1, e_2 \in E$ and $r > 0$, then $\|e_1 - \lambda e_2\| \leq r$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$

implies that $\|e_1 - Te_2\| \le r(1+\varepsilon)$. The set of all ε-multipliers on E is denoted $\mathrm{Mult}_\varepsilon(E)$.

Obviously any multiplier T on E of norm not greater than 1 is an ε-multiplier for all $\varepsilon > 0$, [4, proof of Theorem 3.3]. Also, any ε-multiplier has norm not greater than $1+\varepsilon$. We shall need the following simple propositions.

Proposition 1. *If S and T are ε-multipliers, then so are $-T$ and $(S+T)/2$.*

Proof. The result for $-T$ is obvious. Thus suppose that $\|e_1 - \lambda e_2\| \le r$ for all $|\lambda| \le 1$. We have

$$\|e_1 - [(S+T)/2]e_2\| \le \tfrac{1}{2}\|e_1 - Se_2\| + \tfrac{1}{2}\|e_1 - Te_2\| \le 2\cdot\tfrac{1}{2}r(1+\varepsilon),$$

so that $(S+T)/2$ is an ε-multiplier.

We note, for future reference, that if T is an ε_0-multiplier then it is an ε-multiplier for any $\varepsilon > \varepsilon_0$.

Proposition 2. *Let S be an isomorphism of E_1 onto E_2 with $\|S\| \le 1+\tau$ and $\|S^{-1}\| \le 1+\tau$ for some $\tau > 0$, and let ε be defined by $1+\varepsilon = (1+\tau)^2$. If T is a multiplier on E_1 with $\|T\| \le 1$ and if $\hat{T} := STS^{-1}$ then $\hat{T}$ is an ε-multiplier on E_2.*

Proof. Given $e_1, e_2 \in E_2$ suppose that $\|e_1 - \lambda e_2\| \le r$ for all $\lambda \in \mathbf{K}$ with $|\lambda| \le \|T\|$. Then $\|S^{-1}e_1 - \lambda S^{-1}e_2\| \le r(1+\tau)$ so that, since T is a multiplier, $\|S^{-1}e_1 - TS^{-1}e_2\| \le r(1+\tau)$. Hence $\|e_1 - \hat{T}e_2\| = \|SS^{-1}e_1 - STS^{-1}e_2\| \le r(1+\tau)^2 = r(1+\varepsilon)$.

3. Banach spaces E with $\mathrm{Mult}(E)$ geneologically trivial

Definition 2. Given the Banach space E we will say that $\mathrm{Mult}(E)$ is *geneologically trivial* if for every $\eta > 0$ there exists an $\varepsilon > 0$, $\varepsilon = \varepsilon(\eta, E)$ such that if $T \in \mathscr{B}(E)$ is an ε-multiplier then there exists $\lambda \in \mathbf{K}$ with $\|T - \lambda I\| \le \eta$.

Proposition 3. *Let E be a Banach space and let $\mathscr{F}$ by any free ultrafilter on the integers. Then $\mathrm{Mult}(E)$ is geneologically trivial if $\mathrm{Mult}((E)_{\mathscr{F}})$ is trivial—i.e. consists only of multiples of the identity operator.*

Proof. Suppose that $\mathrm{Mult}((E)_{\mathscr{F}})$ is trivial. If $\mathrm{Mult}(E)$ were not geneologically trivial there would exist an $\eta_0 > 0$ and a sequence of $(1/n)$-multipliers $T_n \in \mathscr{B}(E)$ such that for all $\lambda \in \mathbf{K}$, $\|T_n - \lambda I\| > \eta_0$. Then $T := (T_n)_{\mathscr{F}}$ would be an operator on $(E)_{\mathscr{F}}$ of norm not greater than 1 which is also a multiplier. For suppose that $e_n, v_n \in E$, and $\|(e_n)_{\mathscr{F}} - \lambda(v_n)_{\mathscr{F}}\| = \|(e_n - \lambda v_n)_{\mathscr{F}}\| \le r$ for all $\lambda \in \mathbf{K}$ with $|\lambda| \le 1$. Then for each $k = 1, 2, \ldots$ there exists a set A_k of the filter $\mathscr{F}$ such that if $n \in A_k$ then $\|e_n - \lambda v_n\| \le r(1 + 1/k)$ for $|\lambda| \le 1$ and hence $\|e_n - T_n v_n\| \le r(1 + 1/k)(1 + 1/n)$. It follows that $\|(e_n)_{\mathscr{F}} - T(v_n)_{\mathscr{F}}\| = \lim_{\mathscr{F}} \|e_n - T_n v_n\| \le r$, which proves our claim concerning T.

Since $\mathrm{Mult}((E)_{\mathscr{F}})$ is trivial, there is a $\lambda \in \mathbf{K}$ such that $(T_n - \lambda I)_{\mathscr{F}} = 0$. But for each n there exists an $e_n \in E$ with $\|e_n\| = 1$ and $\|(T_n - \lambda I)e_n\| > \eta_0$.

Thus the element $(e_n)_{\mathscr{F}}$ of $(E)_{\mathscr{F}}$ has norm one and $\|(T_n - \lambda I)_{\mathscr{F}}\| \geq \|(T_n - \lambda I)_{\mathscr{F}}(e_n)_{\mathscr{F}}\| = \lim_{\mathscr{F}} \|(T_n - \lambda I)e_n\| \geq \eta_0$, and this contradiction concludes the proof.

Throughout the next section we will be concerned with Banach duals E which are such that $\mathrm{Mult}(E)$ is geneologically trivial. We wish to observe, via the following two propositions, that the class of such spaces is large enough to be interesting.

Proposition 4. *If E is a uniformly convex or a uniformly smooth Banach space, then* $\mathrm{Mult}(E)$ *is geneologically trivial.*

Proof. In view of Proposition 3 it suffices to show that if E is uniformly convex, (uniformly smooth), then so is $(E)_{\mathscr{F}}$ [4, Proposition 5.1]. This fact is doubtless known. We give the easy proof for uniformly convex spaces. The proof for uniformly smooth spaces is analogous.

Thus suppose that E is uniformly convex. That is, given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $e, v \in E$, $\|e\| < 1$, $\|v\| < 1$ and $\|e - v\| > \varepsilon$ then $\|e + v\| \leq 2 - 2\delta(\varepsilon)$. Hence assume $\varepsilon > 0$ is given and $(e_n)_{\mathscr{F}}$, $(v_n)_{\mathscr{F}}$ are elements of $(E)_{\mathscr{F}}$ with $\|(e_n)_{\mathscr{F}}\| < 1$, $\|(v_n)_{\mathscr{F}}\| < 1$ and $\|(e_n)_{\mathscr{F}} - (v_n)_{\mathscr{F}}\| > \varepsilon$. Then there is a set A in $\mathscr{F}$ such that for $n \in A$ one has $\|e_n\| < 1$, $\|v_n\| < 1$, and $\|e_n - v_n\| > \varepsilon$ so that $\|e_n + v_n\| \leq 2 - 2\delta(\varepsilon)$. Hence $\|(e_n)_{\mathscr{F}} + (v_n)_{\mathscr{F}}\| = \lim_{\mathscr{F}} \|e_n + v_n\| \leq 2 - 2\delta(\varepsilon)$.

Recall that if $1 \leq p < \infty$, an L^p-projection on a Banach space E is a projection $Q : E \to E$ such that

$$\|e\|^p = \|Qe\|^p + \|e - Qe\|^p$$

for $e \in E$.

Proposition 5. *Let E be a Banach space and let $Q : E \to E$ be a nontrivial L^p-projection for some p with $1 \leq p < \infty$. (If $p = 1$ we assume that* $\dim(E) > 2$.) *Then* $\mathrm{Mult}(E)$ *is geneologically trivial.*

Proof. Again, by Proposition 3 it suffices to show that $\mathrm{Mult}((E)_{\mathscr{F}})$ is trivial. If we set $Q_n = Q$ for all n then $\hat{Q} := (Q_n)_{\mathscr{F}}$ is a nontrivial L^p-projection on $(E)_{\mathscr{F}}$. Hence [8, p. 10] $\hat{Q}^{**}$ is a nontrivial L^p-projection on $(E)_{\mathscr{F}}^{**}$. If $(E)_{\mathscr{F}}$ were to admit a nontrivial mutliplier T, then [5, p. 26] T^{**} would be a nontrivial multiplier on $(E)_{\mathscr{F}}^{**}$ so thay by [4, Theorem 5.9] $(E)_{\mathscr{F}}^{**}$ would admit a nontrivial L^∞-projection. But by [3, Theorem 3.5] this is impossible. Hence $\mathrm{Mult}((E)_{\mathscr{F}})$ is trivial and we are done.

4. Isomorphisms of spaces of vector functions

Lemma 1. *Let X be a compact Hausdorff space and E^* a Banach dual such that* $\mathrm{Mult}(E^*)$ *is geneologically trivial. Given $\eta > 0$ let $\varepsilon = \varepsilon(\eta, E^*)$ be related to η as in Definition 2. If then $T : C(X, (E^*, \sigma^*)) \to C(X, (E^*, \sigma^*))$ is an ε-multiplier there is a $g \in C(X)$ with $\|g\|_\infty \leq \|T\|$ such that $\|T(\mathbf{e}^*) - g \cdot e^*\|_\infty \leq 2\eta\|e^*\|$ for all $e^* \in E^*$.*

Proof. Fix $x \in X$. We know that if $e_1^*, e_2^* \in E^*$ and if $\|e_1^* - \lambda e_2^*\| \le r$ for all $\lambda \in \mathbf{K}$, $|\lambda| \le 1$ then

$$\|\mathbf{e}_1^* - \lambda \mathbf{e}_2^*\|_\infty \le r \quad \text{for such } \lambda \text{ so that}$$
$$\|\mathbf{e}_1^* - T(\mathbf{e}_2^*)\|_\infty \le r(1+\varepsilon).$$

Define $S_x : E^* \to E^*$ by $S_x(e^*) = (T(\mathbf{e}^*))(x)$. Thus if $e_1^*, e_3^* \in E^*$ and $\|e_1^* - \lambda e_2^*\| \le r$ for all $|\lambda| \le 1$ we have

$$\begin{aligned}\|e_1^* - S_x(e_2^*)\| &= \|\mathbf{e}_1^*(x) - (T(\mathbf{e}_2^*))(x)\| \\ &\le \|\mathbf{e}_1^* - T(\mathbf{e}_2^*)\|_\infty \le r(1+\varepsilon)\end{aligned}$$

so that S_x is indeed an ε-multiplier on E^* and, obviously, $\|S_x\| \le \|T\|$.

By Definition 2 there exists a $\lambda_x \in \mathbf{K}$ such that

$$\|S_x(e^*) - \lambda_x e^*\| = \|(T(\mathbf{e}^*))(x) - \lambda_x e^*\| \le \eta \|e^*\| \tag{1}$$

for $e^* \in E^*$. Thus fix an $e_0 \in E$ (the predual of E^*) with $\|e_0\| = 1$ and take an $e_0^* \in E^*$ with $\|e_0^*\| = 1$ such that $\langle e_0, e_0^* \rangle = 1$. We have

$$|\langle e_0, (T(\mathbf{e}_0^*))(x)\rangle - \lambda_x| = |\langle e_0, (T(\mathbf{e}_0^*))(x)\rangle - \lambda_x \langle e_0, e_0^*\rangle| \le \eta.$$

Hence, for every $e^* \in E^*$,

$$\|\langle e_0, (T(\mathbf{e}_0^*))(x)\rangle e^* - \lambda_x e^*\| \le \eta \|e^*\|. \tag{2}$$

Thus if $e^* \in E^*$ we have

$$\begin{aligned}&\|(T(\mathbf{e}^*))(x) - \langle e_0, (T(\mathbf{e}_0^*))(x)\rangle e^*\| \\ &\quad \le \|(T(\mathbf{e}^*))(x) - \lambda_x e^*\| + \|\lambda_x e^* - \langle e_0, (T(\mathbf{e}_0^*))(x) e^*\rangle e^*\| \\ &\quad \overset{(1),(2)}{\le} 2\eta \|e^*\|\end{aligned}$$

so that, if we set $g := \langle e_0, (T(\mathbf{e}^*))(\cdot)\rangle$, the proof of the lemma is complete.

Lemma 2. *Let $\eta > 0$ be given and let E be any Banach space. Then there exists an $\varepsilon > 0$, $\varepsilon = \varepsilon(\eta)$, such that if $T : E \to E$ is an ε-multiplier, if $u_0 \in E$, $\|u_0\| \le 1$ with $\|Tu_0\| \le \varepsilon$, and if $v_0 = Tv_1$ where $v_1 \in E$, $\|v_1\| \le 1$ then*

$$\|u_0 + v_0\| \le 1 + \eta.$$

Proof. If the theorem were false then there would exist a number $\eta_0 > 0$, a sequence $\{E_n\}$ of Banach spaces, a sequence $\{T_n\}$ of $(1/n)$-multipliers, $T_n : E_n \to E_n$, and two sequences $\{u_n\}$, $\{v_n'\}$ with $u_n, v_n' \in E_n$ for all n, $\|u_n\| \le 1$, $\|Tu_n\| \le 1/n$, $\|v_n'\| \le 1$ such that if $v_n = T_n v_n'$ then

$$\|u_n + v_n\| > 1 + \eta_0.$$

Let T be the map from $(E_n)_{\mathcal{F}}$ to itself given by $T := (T_n)_{\mathcal{F}}$. Set $u := (u_n)_{\mathcal{F}}$ and $v' := (v_n')_{\mathcal{F}}$. We have $\|u\| \le 1$, $Tu = 0$, $\|v'\| \le 1$ and

$$\|u + Tv'\| \ge 1 + \eta_0 > 1.$$

But T is a multiplier (by the same argument as that used in the proof of Proposition 3) with $\|T\| = \lim_{\mathcal{F}} \|T_n\| \le \lim_{\mathcal{F}}(1 + 1/n) = 1$ and by [6, Lemma 2.2] we have

$$\|u + v\| = \max\{\|u\|, \|v\|\}$$

for u in the kernel of T and v in the range of T. This contradiction concludes the proof of the lemma.

We note that the proof of Lemma 1 shows that there exists a map which associates with each ε-multiplier T on a space $C(X, (E^*, \sigma^*))$, with $\mathrm{Mult}(E^*)$ geneologically trivial, a function $g \in C(X)$ with $\|g\|_\infty \le \|T\|$. We denote this correspondence by writing $g = \rho(T)$. This definition of ρ and the proof of Lemma 1 show that if I is the identity operator on $C(X, (E^*, \sigma^*))$ then $\rho(I) = \mathbf{1}$. Note that if T_1, T_2 and $\alpha T_1 + T_2$ all belong to $\mathrm{Mult}_\varepsilon(C(X, (E^*, \sigma^*)))$ for some $\alpha \in \mathbf{K}$ then $\rho(\alpha T_1 + T_2) = \alpha\rho(T_1) + \rho(T_2)$.

Moreover, given $g \in C(X)$, we will denote by M_g that operator on $C(X, (E^*, \sigma^*))$ which is multiplication by g. Obviously $\|M_g\| = \|g\|_\infty$. Since $\mathrm{Mult}(E^*)$ is geneologically trivial, hence trivial, it follows from [6, Theorem 2.4] and [18, p. 490] that $\mathrm{Mult}(C(X, (E^*, \sigma^*)))$ is precisely the set $\{M_g : g \in C(X)\}$.

Proposition 6. *If $T \in \mathrm{Mult}(E)$ and $\|T\| \le 1 + \varepsilon$ then T is an ε-multiplier on E.*

Proof. Suppose that $e_1, e_2 \in E$ and $r > 0$ are such that for all scalars λ with $|\lambda| \le 1$ we have $\|e_1 - \lambda e_2\| \le r$. Then by setting $\lambda = \pm 1$ and using the triangle inequality we have $\|e_2\| \le r$. Since $T \in \mathrm{Mult}(E)$ we have $T/(1+\varepsilon) \in \mathrm{Mult}(E)$ and $\|T/(1 + \varepsilon)\| \le 1$ so that

$$\begin{aligned}\|e_1 - Te_2\| &\le \|e_1 - [T/(1+\varepsilon)]e_2\| + \|e_2\|\,\|T\|[1 - 1/(1+\varepsilon)] \\ &\le r + r(1+\varepsilon)[1 - 1/(1+\varepsilon)] = r(1+\varepsilon).\end{aligned}$$

Lemma 3. *Let X be a compact Hausdorff space and let E^* be a Banach dual with $\mathrm{Mult}(E^*)$ geneologically trivial. Let η be a given positive number. Let ε_1 denote the $\varepsilon(\eta, E^*)$ of Definition 2 and let ε_2 denote the $\varepsilon(\eta)$ of Lemma 2. Set $\varepsilon_0 = \varepsilon_0(\eta, E^*) := \min\{\varepsilon_2(\eta), \varepsilon_1(\varepsilon_2(\eta), E^*)\}$. Then if T is an ε_0-multiplier on $C(X, (E^*, \sigma^*))$ we have*

$$\|T - M_{\rho(T)}\| \le 2\eta.$$

Proof. Let T be a nonzero ε_0-multiplier. Set

$$\hat{T} := \tfrac{1}{2}(T - M_{\rho(T)}).$$

Since T is an $\varepsilon(\varepsilon_2(\eta), E^*)$-multiplier, by Lemma 1, for any $e^* \in E^*$ we have

$$\|\hat{T}(\mathbf{e}^*)\|_\infty \le \varepsilon_2(\eta)\|e^*\|. \tag{3}$$

Let F be any element of $C(X, (E^*, \sigma^*))$ with $\|F\|_\infty \le 1$. We have

$$\sup\{\|\hat{T}(F) + \mathbf{e}^*\|_\infty : e^* \in E^*, \|e^*\| \le 1\} = 1 + \|\hat{T}(F)\|_\infty.$$

On the other hand, by Propositions 1 and 6 and our choice of ε_0, $\hat{T}$ is an $\varepsilon_2(\eta)$-multiplier so that by (3) and Lemma 2, for any $e^* \in E^*$ with $\|e^*\| \le 1$ we have

$$\|\hat{T}(F) + \mathbf{e}^*\|_\infty \le 1 + \eta .$$

Hence $\|\hat{T}(F)\|_\infty \le \eta$ so that $\|\hat{T}\| \le \eta$ and we are done.

Theorem. *Let X_i be compact Hausdorff spaces and E_i^* Banach duals with* $\mathrm{Mult}(E_i^*)$ *geneologically trivial for $i = 1, 2$. Then there is a positive number ε such that the existence of a surjective isomorphism $S : C(X_1, (E_1^*, \sigma^*)) \to C(X_2, (E_2^*, \sigma^*))$ with $\|S\|\,\|S^{-1}\| < 1 + \varepsilon$ implies that X_1 and X_2 are homeomorphic.*

Proof. First let η be a real number with $0 < \eta < \frac{1}{6}$ and, for $i = 1, 2$, choose $\varepsilon_0(\eta, E_i^*)$ as in Lemma 3. Then let ε be a positive number satisfying $\varepsilon \le \min\{\varepsilon_0(\eta, E_1^*),\ \varepsilon_0(\eta, E_2^*)\}$ and such that

$$(1 + \varepsilon)^2(1 + 2\eta) < \tfrac{4}{3} . \tag{4}$$

In order to facilitate the arguments that follow it will be desirable to have a symmetric relationship between S and S^{-1}. Thus, defining τ by $(1 + \tau)^2 = 1 + \varepsilon$ and replacing S, if necessary, by a suitable scalar multiple we may assume that

$$\frac{1}{1+\tau}\|F\|_\infty \le \|SF\|_\infty \le (1 + \tau)\|F\|_\infty$$

for $F \in C(X_1, (E_1^*, \sigma^*))$, and consequently that $\|S\| \le 1 + \tau$, $\|S^{-1}\| \le 1 + \tau$.

We let ρ be the map from the set of ε-multipliers on $C(X_2, (E_2^*, \sigma^*))$ to $C(X_2)$ which appears in Lemma 3, and note that if $f \in C(X_1)$ and $\|f\|_\infty \le 1$ then, by Proposition 2, $S \circ M_f \circ S^{-1}$ is an ε-multiplier on $C(X_2, (E_2^*, \sigma^*))$. We may thus define a map Φ_0 from the unit ball of $C(X_1)$ to $C(X_2)$ by

$$\Phi_0(f) = \rho(S \circ M_f \circ S^{-1}), \qquad \text{for} \quad f \in C(X_1), \quad \|f\|_\infty \le 1 .$$

If f_1, f_2 and $\alpha f_1 + f_2$ (some $\alpha \in \mathbf{K}$) are all elements of $C(X_1)$ of norm less than or equal to 1, so that by Proposition 2, $S \circ M_{f_1} \circ S^{-1}$, $S \circ M_{f_2} \circ S^{-1}$, and $S \circ M_{\alpha f_1 + f_2} \circ S^{-1}$ are all ε-multipliers on $C(X_2, (E_2^*, \sigma^*))$ then, as noted following the proof of Lemma 2, $\Phi_0(\alpha f_1 + f_2) = \alpha\Phi_0(f_1) + \Phi_0(f_2)$. Thus given $f \in C(X_1)$, take any $R_1 \ge \|f\|_\infty$ and consider $R_1 \cdot \Phi_0(f/R_1)$. If $R_2 > R_1 \ge \|f\|_\infty$ then $R_2 = R_1 \cdot R$ for some $R > 1$ and $R_2 \cdot \Phi_0(f/R_2) = R_1 \cdot R \cdot \Phi_0(f/R_1 \cdot R) = R_1 \cdot R \cdot (1/R) \cdot \Phi_0(f/R_1) = R_1 \cdot \Phi_0(f/R_1)$. Hence if we denote by $\lim_{R\to\infty} R \cdot \Phi_0(f/R)$ the common value of $R_1 \cdot \Phi_0(f/R_1)$ for all $R_1 \ge \|f\|_\infty$, then $\Phi(f) := \lim_{R\to\infty} R \cdot \Phi_0(f/R)$ is a linear map from $C(X_1)$ to $C(X_2)$ which agrees with Φ_0 on the unit ball of $C(X_1)$. (Equivalently, $\Phi(f) = \|f\|_\infty \Phi_0(f/\|f\|_\infty)$, for $f \ne 0$.) Now if $0 \ne f \in C(X_1)$ set $f_1 = f/\|f\|_\infty$. Then by Lemma 3 $\|S \circ M_{f_1} \circ S^{-1} - M_{\Phi(f_1)}\| \le 2\eta$ so that

$$\|S \circ M_f \circ S^{-1} - M_{\Phi(f)}\| \le 2\eta\|f\|_\infty , \qquad f \in C(X_1) . \tag{5}$$

Hence we have

$$\begin{aligned}| \|S \circ M_f \circ S^{-1}\| - \|\Phi(f)\|_\infty| &\le \|S \circ M_f \circ S^{-1} - M_{\Phi(f)}\| \\ &\le 2\eta\|f\|_\infty, \qquad f \in C(X_1),\end{aligned}$$

and it is clear that

$$\|f\|_\infty/(1+\varepsilon) < \|S \circ M_f \circ S^{-1}\| \le (1+\varepsilon)\|f\|_\infty$$

for $f \in C(X_1)$. It follows that

$$(6)\quad [1/(1+\varepsilon) - 2\eta]\|f\|_\infty \le \|\Phi(f)\|_\infty \le [1+\varepsilon+2\eta]\|f\|_\infty, \qquad f \in C(X_1).$$

Let Ψ_0 be the corresponding map from the unit ball of $C(X_2)$ to $C(X_1)$ given by $\Psi_0(g) = \rho(S^{-1} \circ M_g \circ S)$ for $g \in C(X_2)$, $\|g\|_\infty \le 1$. By symmetry we have

$$(7)\qquad \|S^{-1} \circ M_g \circ S - M_{\Psi_0(g)}\| \le 2\eta\|g\|_\infty, \qquad \|g\|_\infty \le 1,$$

and if Ψ corresponds to Ψ_0 as Φ corresponds to Φ_0, then

$$(8)\qquad \|\Psi\| \le 1+\varepsilon+2\eta,$$

and (7) holds with Ψ replacing Ψ_0 for all $g \in C(X_2)$. Thus for any $g \in C(X_2)$ with $\|g\|_\infty \le 1$ we have

$$\begin{aligned}\|\Phi(\Psi(g)) - g\|_\infty &= \|M_{\Phi(\Psi(g))} - M_g\| \\ &= \|M_{\Phi(\Psi(g))} - S \circ (S^{-1} \circ M_g \circ S) \circ S^{-1}\| \\ &\le \|M_{\Phi(\Psi(g))} - S \circ M_{\Psi(g)} \circ S^{-1}\| \\ &\quad + \|S \circ M_{\Psi(g)} \circ S^{-1} - S \circ (S^{-1} \circ M_g \circ S) \circ S^{-1}\| \\ &\overset{(5),(7)}{\le} 2\eta\|\Psi(g)\|_\infty + \|S\|2\eta\|S^{-1}\| \\ &\overset{(8)}{\le} 2\eta[1+\varepsilon+2\eta] + 2\eta(1+\varepsilon)^2 \\ &< 2\eta(1+\varepsilon)[1+\varepsilon+2\eta(1+\varepsilon)] + 2\eta(1+\varepsilon)^2 \\ &= 4\eta[1+\eta](1+\varepsilon)^2.\end{aligned}$$

As (4) implies that

$$(9)\qquad 2\eta(1+\varepsilon)^2 < \tfrac{1}{3}$$

the condition $\eta < \frac{1}{6}(< \frac{1}{2})$ gives $\|\Phi(\Psi(g)) - g\| < 1$, and thus, by the Riesz lemma, Φ is surjective. And since (9) gives

$$(10)\qquad 1 - 2\eta(1+\varepsilon) > 1 - 2\eta(1+\varepsilon)^2 > \tfrac{2}{3}$$

it now follows from the inequality on the left in (6) that Φ is injective. Thus Φ is an isomorphism mapping $C(X_1)$ onto $C(X_2)$ which, by (6), satisfies

$$\|\Phi\|\,\|\Phi^{-1}\| \le \frac{(1+\varepsilon)^2 + 2\eta(1+\varepsilon)}{1-2\eta(1+\varepsilon)} < \frac{(1+\varepsilon)^2[1+2\eta]}{1-2\eta(1+\varepsilon)^2}.$$

Since by (4) the numerator in this last expression is less than $\frac{4}{3}$ and by (10) the denominator is greater than $\frac{2}{3}$, we have $\|\Phi\| \|\Phi^{-1}\| < 2$ so that X_1 and X_2 are homeomorphic [1, 9, 10].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CALIFORNIA 93106

DEPARTMENT OF MATHEMATICS AND STATISTICS, SOUTHERN ILLINOIS UNIVERSITY AT EDWARDSVILLE, EDWARDSVILLE, ILLINOIS 62026