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STABLE RANK OF SOME CROSSED PRODUCT C^*-ALGEBRAS

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ABSTRACT. Let $C(X) \times_T Z$ be the crossed product associated to a dynamical system (X, T). We give a necessary and sufficient condition for $C(X) \times_T Z$ to have a dense set of invertible elements. When X is zero-dimensional, we obtain more equivalent conditions which involve the isomorphism between the K-groups of $C(X) \times_T Z$ and C^*-algebras defined by some T-invariant closed subsets of X. As an application, we show that these conditions are not satisfied by most subshifts and all nontrivial irreducible Markov shifts. When (X, T) is indecomposable, an equivalent condition is that the intersection of all T-invariant nonempty closed subsets of X is nonempty.

1. INTRODUCTION

Given a unital C^*-algebra A, let $\mathrm{Lg}_n(A)$ be the set of n-tuples in A^n which generates A as a left ideal. The topological stable rank of A is defined (Rieffel [15]) as the smallest integer n such that $\mathrm{Lg}_n(A)$ is dense in A^n. If no such n exists, the topological stable rank of A is defined to be ∞. For simplicity, we will just call this the stable rank of A, $\mathrm{sr}(A)$. If A does not have a unit, then $\mathrm{sr}(A)$ is defined to be $\mathrm{sr}(\widetilde{A})$, where $\widetilde{A}$ is the C^*-algebra obtained from A by adjoining a unit [8]. One of the reasons for studying stable rank is to obtain cancellation theorems for the classification of projective modules over A (e.g. Rieffel [16], Sheu [19]). Thus, given a C^*-algebra A, one would like to determine $\mathrm{sr}(A)$. In particular, $\mathrm{sr}(A) = 1$ if and only if the invertible elements are dense in A. This case has attracted a lot of attention [2, 6, 7, 12, 15, 17, 18]. One of the nice properties of these C^*-algebras is that they all have cancellation for projections [1, 6.4.1]. In this note, we will study the stable rank of the crossed product associated to some dynamical systems.

A (dynamical) system (X, T) consists of a compact space X and a homeomorphism T on X. Given a system (X, T), we have an action of the integers Z on $C(X)$, the C^*-algebra of complex continuous functions on X. This gives a crossed product $C(X) \times_T Z$ [8], which is a C^*-algebra generated by

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$C(X)$ and a unitary U satisfying $UfU^* = f \circ T^{-1}$ for $f \in C(X)$. If Y is a nonempty T-invariant closed subset of X, then we have a system (Y, T). By restricting the functions in $C(X)$ to Y, we have a C^*-homomorphism π from $C(X) \times_T Z$ onto $C(Y) \times_T Z$. Let I_Y be the kernel of π. Our first result is that $\mathrm{sr}(C(X) \times_T Z) = 1$ if and only if $\mathrm{sr}(I_Y) = \mathrm{sr}(C(Y) \times_T Z) = 1$ and the homomorphism $\pi_*: K_1(C(X) \times_T Z) \to K_1(C(Y) \times_T Z)$ is onto. Then we restrict our attention to systems (X, T) where X is zero-dimensional. A compact metrizable space is said to be zero-dimensional if the topology on X has a basis of sets which are both closed and open (clopen). For such systems, $\mathrm{sr}(C(X) \times_T Z)$ is either 1 or 2 (Rieffel [15, Theorem 7.1]). By computing the K_1-groups explicitly, we derive some necessary conditions on (X, T) for $\mathrm{sr}(C(X) \times_T Z) = 1$. An application of this result shows that for most subshifts and all nontrivial irreducible Markov shifts [4], $\mathrm{sr}(C(X) \times_T Z) = 2$. A system (X, T) is said to be minimal if X contains no nontrivial T-invariant closed subsets. In [12], Putnam proved that if the zero-dimensional system (X, T) is minimal and X has no isolated points, then $\mathrm{sr}(C(X) \times_T Z) = 1$. A key step in his proof is that for every nonempty closed subset Y of X, the C^*-subalgebra A_Y of $C(X) \times_T Z$ generated by $C(X)$ and $\{Uf : f \in C(X), f(y) = 0$ for all $y \in Y\}$ is an AF-algebra—i.e., A_Y is the closure of an increasing sequence of finite-dimensional subalgebras [5]. This result has been generalized to the following:

Proposition 1.1 [11, Theorem 2.2]. *Given any zero-dimensional system (X, T) and a nonempty closed subset Y of X, the subalgebra A_Y is AF if and only if $\bigcup_{n \in Z} T^n(W) = X$ for every clopen subset W containing Y.*

We will use $D(X, T)$ to denote the set of closed subsets Y of X satisfying the condition in the above proposition. Suppose $Y \in D(X, T)$ is T-invariant. Theorem 3.1 gives three conditions equivalent to $\mathrm{sr}(C(X) \times_T Z) = 1$, one of which is that $\mathrm{sr}(C(Y) \times_T Z) = 1$ and every T-invariant clopen subset of Y is the intersection of Y and a T-invariant clopen subset of X. Let $E(X, T)$ be the set of minimal (in the sense of inclusion) elements in $D(X, T)$. Suppose $Y \in E(X, T)$ is T-invariant. Let i be the embedding of A_Y into $C(X) \times_T Z$. Then Theorem 3.4 shows that $\mathrm{sr}(C(X) \times_T Z) = 1$ if and only if $i_*: K_0(A_Y) \to K_0(C(X) \times_T Z)$ is an isomorphism. A system (X, T) is said to be indecomposable if X and $\varnothing$ are the only T-invariant clopen subsets of X. In §4, we prove that if (X, T) is an indecomposable zero-dimensional system, then $\mathrm{sr}(C(X) \times_T Z) = 1$ if and only if the intersection of all T-invariant nonempty closed subsets of X is nonempty. We conclude with some remarks and an example in connection with a result of Pimsner [9].

We will use Blackadar [1], Effros [5], and Pedersen [8] for our references on K-theory, AF-algebras and C^*-algebras.

Theorem 4.1 has also been proven by Putnam in a revised version of [13], which we received after this paper had been submitted.

2. Stable rank of crossed products

We start with a result of G. Nagy (Nistor [7, Lemma 3]):

Lemma 2.1. *Let* $0 \to I \to A \to B \to 0$ *be an exact sequence of* C^**-algebras such that* $\mathrm{sr}(I) = \mathrm{sr}(B) = 1$. *Then* $\mathrm{sr}(A) = 1$ *if and only if the index morphism* $\delta : K_1(B) \to K_0(I)$ *is zero.*

Lemma 2.2. *Let* $0 \to I \xrightarrow{i} A \xrightarrow{\pi} B \to 0$ *be an exact sequence of* C^**-algebras. Then* $\mathrm{sr}(A) = 1$ *if and only if* $\mathrm{sr}(B) = \mathrm{sr}(I) = 1$ *and* $\pi_*: K_1(A) \to K_1(B)$ *is onto.*

Proof. From $0 \to I \xrightarrow{i} A \xrightarrow{\pi} B \to 0$, we have a six-term exact sequence of K-groups [1]. So the result follows from the exactness at $K_1(B)$:

$$\to K_1(A) \xrightarrow{\pi_*} K_1(B) \xrightarrow{\delta} K_0(I) \to$$

and Lemma 2.1. □

So far, most of the results on determining $\mathrm{sr}(A) = 1$ have been done on simple C^*-algebras A (e.g. [12, 14]). Given a dynamical system (X, T), the crossed product $C(X) \times_T Z$ is not simple if and only if there exists a nonempty T-invariant proper closed subset Y of X. By restricting the action of T and the functions in $C(X)$ on Y, we have a surjective C^*-homomorphism $\pi: C(X) \times_T Z \to C(Y) \times_T Z$. Let I_Y be the kernel of π; then we have an exact sequence of C^*-algebras $0 \to I_Y \to C(X) \times_T Z \to C(Y) \times_T Z \to 0$. Applying Lemma 2.2 to this exact sequence, we have

Theorem 2.3. *Let* (X, T) *be a dynamical system and* Y *a nonempty* T*-invariant closed subset of* X. *Then* $\mathrm{sr}(C(X) \times_T Z) = 1$ *if and only if*

$$\mathrm{sr}(I_Y) = \mathrm{sr}(C(Y) \times_T Z) = 1$$

and $\pi_*: K_1(C(X) \times_T Z) \to K_1(C(Y) \times_T Z)$ *is onto.*

Given a zero-dimensional system (X, T), let $C(X, Z)$ be the group of integer-valued continuous functions on X and $C^T(X, Z)$ the T-invariant functions in $C(X, Z)$. Suppose Y is a nonempty closed subset of X. Define $\Phi_Y: C(X, Z) \to C(Y, Z)$ by $\Phi_Y(f) = f|_Y$, the restriction of f to Y.

Lemma 2.4. *Let* (X, T) *be a zero-dimensional dynamical system and* Y *a nonempty* T*-invariant closed subset of* X. *Then* $K_1(C(X) \times_T Z) \simeq C^T(X, Z)$, $K_1(C(Y) \times_T Z) \simeq C^T(Y, Z)$ *and the map* $\pi_*: K_1(C(X) \times_T Z) \to K_1(C(Y) \times_T Z)$ *is given by* $\Phi_Y: C^T(X, Z) \to C^T(Y, Z)$.

Proof. We compute $K_1(C(X) \times_T Z)$ by the Pimsner and Voiculescu six-term exact sequence [10]:

$$\begin{array}{ccccc}
K_1(C(X)) & \xrightarrow{\mathrm{id}_* - T_*} & K_1(C(X)) & \xrightarrow{i_*} & K_1(C(X) \times_T Z) \\
\uparrow & & & & \downarrow \\
K_0(C(X) \times_T Z) & \xleftarrow{i_*} & K_0(C(X)) & \xleftarrow{\mathrm{id}_* - T_*} & K_0(C(X))
\end{array}$$

Since X is zero-dimensional, $K_1(C(X)) = 0$. Hence, the map $K_1(C(X) \times_T Z) \to K_0(C(X))$ is always one-to-one. Also, $K_0(C(X))$ is isomorphic to $C(X, Z)$, the integer-valued continuous functions on X. Thus, $K_1(C(X) \times_T Z)$ is isomorphic to the kernel of $\mathrm{id}_* - T_*: K_0(C(X)) \to K_0(C(X))$, which is precisely $C^T(X, Z)$ (see [10] for details on the homomorphisms in the exact sequence). For each $f \in C^T(X, Z)$, there exist integers n_i, $1 \le i \le k$, and a clopen partition $\{O_i: 1 \le i \le k\}$ of X such that each O_i is T-invariant and $f = \sum_{i=1}^k n_i \chi_{O_i}$, where χ_O denotes the characteristic function on O. Since all O_i are T-invariant, $\sum_{i=1}^k U^{n_i} \chi_O$ is a unitary of $C(X) \times_T Z$. An analysis of the connecting homomorphisms in the proof in [10] shows that $\sum_{i=1}^k n_i \chi_{O_i} \to \ulcorner \sum_{i=1}^k U^{n_i} \chi_{O_i} \urcorner$ gives an isomorphism of $C^T(X, Z)$ and $K_1(C(X) \times_T Z)$.

Similarly, $K_1(C(Y) \times_T Z) \simeq C^T(Y, Z)$ and the result follows. □

Remark 2.5. Under the conditions in Lemma 2.4, we note that the map $\Phi_Y: C^T(X, Z) \to C^T(Y, Z)$ is onto if and only if for every T-invariant clopen subset Q of Y, there exists a T-invariant clopen subset O of X such that $Q = O \cap Y$.

Corollary 2.6. *Suppose (X, T) is a zero-dimensional system with no nontrivial T-invariant clopen subsets. If X contains two disjoint nonempty T-invariant closed subsets, then* $\mathrm{sr}(C(X) \times_T Z) = 2$.

Example 2.7. Let (X, T) be a zero-dimensional system which contains a point with dense orbit and two periodic points x_1, x_2 with disjoint orbits. Then the conditions in Corollary 2.6 are satisfied and $\mathrm{sr}(C(X) \times_T Z) = 2$. Hence, for most subshifts and all nontrivial irreducible Markov shifts [4] (X, T), $\mathrm{sr}(C(X) \times_T Z) = 2$.

3. Subsets in $D(X,T)$ and $E(X,T)$

Throughout this section, (X, T) will denote a zero-dimensional dynamical system. For each nonempty closed subset Y of X, A_Y is the subalgebra of $C(X) \times_T Z$ generated by $C(X)$ and $\{Uf: f \in C(X), f(y) = 0 \text{ for all } y \in Y\}$. Let $D(X, T)$ be the set of closed subsets Y of X such that $\bigcup_{n \in Z} T^n(W) = X$ for every clopen subset W containing Y. By Proposition 1.1, A_Y is an AF subalgebra if and only if $Y \in D(X, T)$. Let $E(X, T)$ be the set of minimal (in the sense of inclusion) elements in $D(X, T)$ [11].

Theorem 3.1. *Let Y be a T-invariant subset in $D(X, T)$ and $\pi: C(X) \times_T Z \to C(Y) \times_T Z$ as defined in §2. Then the following are equivalent:*

(1) $\mathrm{sr}(C(X) \times_T Z) = 1$.
(2) $\mathrm{sr}(C(Y) \times_T Z) = 1$ *and* $\pi_*: K_1(C(X) \times_T Z) \to K_1(C(Y) \times_T Z)$ *is an isomorphism.*
(3) $\mathrm{sr}(C(Y) \times_T Z) = 1$ *and* $\Phi_Y: C^T(X, Z) \to C^T(Y, Z)$ *is an isomorphism.*

(4) $\mathrm{sr}(C(Y)\times_T Z) = 1$ *and for each* T*-invariant clopen subset* Q *of* Y *there exists a* T*-invariant clopen subset* O *of* X *such that* $Q = O \cap Y$.

Proof. Let I_Y be the kernel of π, i.e., I_Y is the ideal of $C(X)\times_T Z$ generated by functions in $C(X)$ vanishing in Y. Since Y is T-invariant, I_Y is an ideal of the AF subalgebra A_Y. Thus, I_Y is also AF. So we have $\mathrm{sr}(I_Y) = 1$ and $K_1(I_Y) = 0$. Hence, π_* is one-to-one and the result follows from Theorem 2.3, Lemma 2.4, and Remark 2.5. □

Remark 3.2. In the above theorem, since π_* is always one-to-one, the "isomorphism" conditions in (2) and (3) can be replaced by "onto".

Before proving the next theorem, we need the following generalization of a result of Putnam [13, Theorem 4.1]:

Proposition 3.3. *Let* $Y \in D(X, T)$. *There is an exact sequence*

$$0 \to C^T(X, Z) \xrightarrow{\Phi_Y} C(Y, Z) \xrightarrow{\psi} K_0(A_Y) \xrightarrow{i_*} K_0(C(X)\times_T Z) \to 0.$$

Here i is the embedding of A_Y into $C(X)\times_T Z$. For our application, the definition of ψ is not important. We include it here just for completeness: Given $f \in C(Y, Z)$, we choose $g \in C(X, Z)$ such that $g|_Y = f$. Let i_1 be the embedding of $C(X)$ into A_Y and $i_{1*}: K_0(C(X)) \to K_0(A_Y)$. Identifying $K_0(C(X))$ with $C(X, Z)$, we put $\psi(f) = i_{1*}(g - g\circ T)$. This definition is due to Putnam in [13], where he proved the result for minimal systems (X, T). But, the proof for the general case is essentially the same.

Theorem 3.4. *Let* Y *be a* T*-invariant subset in* $E(X, T)$. *The following conditions are equivalent*:

(1) $\mathrm{sr}(C(X)\times_T Z) = 1$.
(2) $\pi_*: K_1(C(X)\times_T Z) \to K_1(C(Y)\times_T Z)$ *is an isomorphism.*
(3) $\Phi_Y: C^T(X, Z) \to C(Y, Z)$ *is an isomorphism.*
(4) $i_*: K_0(A_Y) \to K_0(C(X)\times_T Z)$ *is an isomorphism.*
(5) *For each clopen subset* Q *of* Y *there exists a* T*-invariant clopen subset* O *of* X *such that* $Q = O\cap Y$.

Proof. First we note that for any $Y \in D(X, T)$, conditions (3), (4) and (5) are always equivalent by Proposition 3.3.

Let $Y \in E(X, T)$ be T-invariant. We are going to prove that $T(y) = y$ for all $y \in Y$. Suppose the contrary that $T(y) \neq y$ for some $y \in Y$. Then we can choose a clopen subset O of X containing y such that $O \cap T(O) = \varnothing$. So, we have that $Y\backslash O$ is a proper closed subset of Y. Let W be a clopen subset of X containing $Y\backslash O$; we have

$$\begin{aligned} T^{-1}(W) &\supseteq T^{-1}(Y\backslash O) \supseteq T^{-1}(Y\cap T(O)) \supseteq Y \cap O \\ &\Rightarrow W \cup T^{-1}(W) \supseteq Y \\ &\Rightarrow \bigcup_{n\in Z} T^n(W) = \bigcup_{n\in Z} T^n(W\cup T^{-1}(W)) = X. \end{aligned}$$

Thus, $Y\backslash O \in D(X, T)$, contradicting $Y \in E(X, T)$.

Since the action of T on Y is the identity, $C(Y) \times_T Z$ is isomorphic to the C^*-tensor product $C(Y) \otimes C(S)$ [8], where S is the unit circle. Since Y is zero-dimensional, $\mathrm{sr}(C(Y) \otimes C(S)) = 1$. So the result follows from Theorem 3.1 because $C^T(Y, Z) = C(Y, Z)$. $\square$

Remark 3.5. If $Y \in E(X, T)$ consists of a single fixed point, then condition (5) and hence all conditions in Theorem 3.4 are obviously satisfied. In Corollary 4.2, we will give a partial converse of this result. Here, we give a class of systems satisfying this condition:

Let T be a continuous strictly increasing function on the unit interval $[0, 1]$ with $T(0) = 0$, $T(1) = 1$ and $T(x) \neq x$ for $0 < x < 1$. Choose a countable T-invariant dense subset S of the open interval $(0, 1)$. For each $0 < s < t < 1$, let $\chi_{[s,t)}$ be the characteristic function on the interval $[s, t)$. Let A be the commutative C^*-algebra generated by $\{\chi_{[s,t)} : s, t \in S\}$ and the constant function 1. Then A is isomorphic to $C(X)$ for a zero-dimensional space X which contains the interval $[0, 1)$ and the action of T extends to X. One checks that 0 is a fixed point and $\{0\} \in E(X, T)$.

The above construction is similar to the one of Cuntz [3, Example 2.5]. Similar examples can also be constructed on higher-dimensional analogues of the unit interval.

4. Indecomposable systems

Given a system (X, T), if X can be decomposed into two disjoint nonempty T-invariant closed subsets X_1 and X_2, then $C(X) \times_T Z$ is isomorphic to the direct sum $\bigoplus_{i=1}^2 C(X_i) \times_T Z$. Hence, $\mathrm{sr}(C(X) \times_T Z) = 1$ if and only if $\mathrm{sr}(C(X_i) \times_T Z) = 1$, for $i = 1, 2$. (X, T) is called indecomposable if no such decomposition exists, i.e., the only T-invariant clopen subsets of X are X and $\varnothing$.

Theorem 4.1. *Let (X, T) be an indecomposable zero-dimensional dynamical system. Then $\mathrm{sr}(C(X) \times_T Z) = 1$ if and only if the intersection of all nonempty T-invariant closed subsets of X is nonempty.*

Proof. Let Y be the intersection of all nonempty T-invariant closed subsets of X.

Suppose $\mathrm{sr}(C(X) \times_T Z) = 1$. Since (X, T) is indecomposable, by Corollary 2.6, the intersection of any two nonempty T-invariant closed subsets of X is nonempty. Thus, by the finite intersection property, Y is nonempty.

Conversely, suppose Y is nonempty. Clearly, Y is a T-invariant closed subset of X. We are going to show that (1) $Y \in D(X, T)$ and (2) the action of T on Y is minimal. Then the result will follow from (4) in Theorem 3.1 because $\mathrm{sr}(C(Y) \times_T Z) = 1$ for a minimal system (Y, T) (Putnam [13]).

To prove (1), let W be a clopen subset containing Y. Then $X \setminus \bigcup_{n \in Z} T^n(W)$ is a T-invariant closed subset disjoint from Y. Thus $X \setminus \bigcup_{n \in Z} T^n(W) = \varnothing$ and $\bigcup_{n \in Z} T^n(W) = X$. Hence, $Y \in D(X, T)$.

To prove (2), for every $y \in Y$, the orbit closure of y is a T-invariant closed subset of Y and hence is equal to Y. □

Corollary 4.2. *Suppose (X, T) is an indecomposable zero-dimensional system such that $\mathrm{sr}(C(X) \times_T Z) = 1$. Then every T-invariant Y in $E(X, T)$ consists of a single point.*

Proof. From the proof of Theorem 3.4, we have that $T(y) = y$ for every $y \in Y$. Thus, Y is nonempty only when it consists of a single point. □

Remark 4.3. Let (X, T) be a (not necessarily zero-dimensional) dynamical system. A point $x \in X$ is said to be pseudo-nonwandering (Pimsner [9]) if for every open cover $\{O_i\}_{i=1}^n$ of X and O_{i_1} containing x, there exist O_{i_k}, $2 \leq k \leq m$, such that $O_{i_j} \cap T^{-1}(O_{i_{j+1}}) \neq \varnothing$ for $2 \leq j < m$ and $O_{i_m} \cap T^{-1}(O_{i_1}) \neq \varnothing$. Let $X(T)$ be the set of all pseudo-nonwandering points in X. Pimsner [9] proved that the following three conditions are equivalent: (1) $X(T) = X$, (2) $C(X) \times_T Z$ contains no nonunitary isometry, and (3) there is a unital imbedding of $C(X) \times_T Z$ into an AF algebra. Since a C^*-algebra with stable rank 1 cannot contain any nonunitary isometry, $X(T) = X$ is a necessary condition for $\mathrm{sr}(C(X) \times_T Z) = 1$ (this connection is communicated to us by Putnam). The following example shows that the condition is not sufficient.

Example 3.7. Let $X = Z \cup \{\infty, -\infty\}$ be the two-point compactification of the integers. Define a homeomorphism $T: X \to X$ by

$$T(\infty) = \infty, \qquad T(-\infty) = -\infty,$$

$$T(x) = \begin{cases} x + 2, & x \in Z, \text{ even}, \\ x - 2, & x \in Z, \text{ odd}. \end{cases}$$

It is straightforward to check that $X(T) = X$ and $Y = \{\infty, -\infty\}$ is a T-invariant subset in $E(X, T)$. Hence, $\mathrm{sr}(C(X) \times_T Z) \neq 1$ by Corollary 4.2.

We note from Example 2.7 that any nontrivial irreducible Markov shift can also serve this purpose.

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