PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 105, Number 4, April 1989

GENERALIZATION OF A RESULT OF BORWEIN AND DITOR

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(Communicated by R. Daniel Mauldin)

ABSTRACT. D. Borwein and S. Z. Ditor have found a measurable subset A of the real line having positive Lebesgue measure and a decreasing sequence (d_n) of reals converging to zero such that, for each x, $x + d_n$ is not in A for infinitely many n; thus answering a question of P. Erdös. It will be shown that the result of Borwein and Ditor can be extended using a general 2-place function in place of plus. Related material is also presented.

1. INTRODUCTION

D. Borwein and S. Z. Ditor [1] have proved the following theorem, answering a question of P. Erdös.

Theorem B and D. (1) *If A is a measurable set of real numbers with $m(A) > 0$ and (d_n) is a sequence converging to zero, then, for almost all $x \in A$, $x+d_n \in A$ for infinitely many n.*

(2) *There is a measurable set A, with $m(A) > 0$ and a monotonic sequence (d_n) of positive reals converging to zero such that, for each x, $x + d_n \notin A$ for infinitely many n.*

The Baire set analogue of (2) is not true. In [2] it is shown that if A is a subset of the real line satisfying the Baire property (see [3]) and (d_n) is a sequence converging to zero, then the set

$$A \backslash \{x \in A : x + d_n \in A \quad \text{for all but finitely many } n\}$$

is of the first Baire category.

Suppose that A is a measurable set of positive reals and (e_n) is a sequence converging to one. It is natural to consider a multiplicative version of the Theorem of Borwein and Ditor. In this paper, we extend the Theorem of Borwein and Ditor to a general result that includes the operations of addition

Received by the editors April 25, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 28A05, 26A21.
The research for this paper was supported by the Scientific Research Fund of Bosna and Herzegovina.

and multiplication as special cases by using general 2-place functions that satisfy appropriate conditions.

2. Results

Our first theorem, an extension of part (2) of the Theorem of Borwein and Ditor, is the main result of this paper.

Theorem 1. *If* $f: R \times R \to R$ *(the real line) is a function satisfying the following conditions*:

(a) *there exists an* $e \in R$ *such that* $f(x, e) = x$ *for every* $x \in R$;
(b) f_x *and* f_y *(partial derivatives) exist and are continuous in a neighborhood of* (x_0, e), *for some* $x_0 \in R$;
(c) $f_y(x_0, e) > 0$,

then there exists a strictly decreasing sequence (e_n) *of real numbers converging to* e *and a measurable subset* A *of* R *with* $m(A) > 0$ *such that, for all* $x \in R$, $f(x, e_n) \notin A$ *for infinitely many* n.

Proof. By (a) and (c), $f_y(x_0, e) = a > 0$ and $f_x(x_0, e) = 1$. Therefore, by (b), there exist $x_1 > x_0$ and $y_1 > e$ such that f_x and f_y exist and are continuous on the closed rectangular body T having corners at (x_0, e), (x_0, y_1), (x_1, y_1), (x_1, e), $f_y > a/2$ and $f_x > 0$ on T. Let $t = y_1 - e$. There exists an integer n_0, $n_0 > 2$ such that $(a/2)(t/2) > (x_1 - x_0)/n_0$. Divide $[x_0, x_1]$ into n_0 abutting closed intervals, each of length $(x_1 - x_0)/n_0$; denote them by $I_1, I_2, \dots, I_{n_0}$. Let $J_1, J_2, \dots, J_{n_0}$ be n_0 open intervals of equal length satisfying:

(i) $J_i \subset I_i$ for $i = 1, 2, \dots, n_0$.
(ii) J_i and I_i have the same right-hand end points for each $i = 1, 2, \dots, n_0$.
(iii) The set $A_1 = [x_0, x_1] \backslash (\bigcup_{i=1}^{n_0} J_i \cup \{x_1\})$ has measure greater than $(0.9)\ (x_1 - x_0)$.

Then, clearly, A_1 can be written as $A_1 = \bigcup_{i=1}^{n_0} H_i$; where the H_i's are closed, disjoint intervals of equal length. Since f_x is bounded on T, by the mean value theorem for derivatives applied to f_x, there exists a $d_1 > 0$ such that if K is a closed subinterval of $[x_0, x_1]$ of length less than d_1 and $e < \bar{e} < y_1$, then the closed interval $f(K, \bar{e}) = \{f(x, \bar{e}): x \in K\}$ has length less than $u_1/2$, where u_1 is the common length of the J_i's.

Let m_1, $m_1 > 2$, be an integer satisfying the inequality $\min(d_1, (a/2)(t/2^2)) > (x_1 - x_0)/m_1 n_0$. Divide each of the closed intervals H_i into m_1 abutting closed intervals of equal length; denote these intervals by $H_{i1}, H_{i2}, \dots, H_{im_1}$.

Suppose K is an interval in the collection $\{H_{ij}: 1 \le i \le n_0, 1 \le j \le m_1\}$, say $K = H_{ij}$. By the mean value theorem for derivatives applied to f_y and the definition of n_0, there exists an e_K, $e < e_K < e + t/2$, such that $L_{J_i} < f(L_K, e_K) < M_{J_i}$, where L_{J_i} and L_K denote the left end points of J_i and

K respectively and M_{J_i} denotes the midpoint of J_i. Furthermore, since the length of K is less than $(x_1 - x_0)/m_1 n_0$, which in turn is less than d_1, it follows that $f(K, e_K)$ has length less than $u_1/2$ and hence $f(K, e_K) \subset J_i$. Of course, for each K, e_K is highly nonunique; namely, if $e_{K'}$ is sufficiently close to e_K than $f(K, e_{K'}) \subset J_i$ still holds.

Therefore, there exist $m_1 n_0$ distinct points $\{e_{1i}\}_{i=1}^{m_1 n_0}$, each strictly between e and $e + t/2$, such that for each $x \in A_1$, there exists a point $e_x \in \{e_{1i}: i = 1, 2, \ldots, m_1 n_0\}$ such that $f(x, e_x) \notin A_1$.

Suppose now that $\{J_{ij}: 1 \le i \le n_0, 1 \le j \le m_1\}$ is a collection of open intervals of equal length satisfying:

(iv) $J_{ij} \subset H_{ij}$ for all $1 \le i \le n_0$ and $1 \le j \le m_1$.

(v) J_{ij} and H_{ij} have the same right-hand end points for all $1 \le i \le n_0$ and $1 \le j \le m_1$.

(vi) $A_2 = \bigcup_{i=1}^{n_0}(H_i \backslash (\bigcup_{j=1}^{m_1} J_{ij} \cup \{b_i\}))$ has measure greater than (0.9) $(x_1 - x_0)$, where b_i is the right end point of H_i for each $1 \le i \le n_0$.

Let u_2 denote the common length of the J_{ij}'s. Again, since f_x is bounded on T, by the mean value theorem there exists a $d_2 > 0$ such that $f(K, \bar{e})$ has length less than $u_2/2$ for every closed subinterval K of $[x_0, x_1]$ of length less than d_2 and every $\bar{e}, e < \bar{e} < y_1$.

Let m_2, $m_2 > 2$ be an integer satisfying the inequality

$$\min(d_2, (a/2)(t/2^3)) > (x_1 - x_0)/(m_2 m_1 n_0).$$

Divide each of the $m_1 n_0$ disjoint closed intervals of equal length making up A_2 into m_2 abutting closed intervals of equal length. Denote the m_2 abutting closed intervals making up H_{ij} by $H_{ij1}, H_{ij2}, \ldots, H_{ijm_2}$.

Suppose K is an interval in the collection $\{H_{ijk}: 1 \le i \le n_0, 1 \le j \le m_1, 1 \le k \le m_2\}$, say $K = H_{ijk}$. By the mean value theorem applied to f_y and the definition of m_1, there exists an e_K, $e < e_K < e + t/2^2$, such that $L_{J_{ij}} < f(L_K, e_K) < M_{J_{ij}}$, where $L_{J_{ij}}$ and L_K are the left end points of J_{ij} and K respectively and $M_{J_{ij}}$ is the midpoint of J_{ij}. Furthermore, since the length of K is less than $(x_1 - x_0)/(m_2 m_1 n_0)$ which in turn is less than d_2, it follows that $f(K, e_K)$ has length less than $u_2/2$ and hence $f(K, e_K) \subset J_{ij}$.

Therefore, there exist $m_2 m_1 n_0$ distinct numbers $\{e_{2i}\}_{i=1}^{m_2 m_1 n_0}$, each different from the points in $\{e_{1i}\}_{i=1}^{m_1 n_0}$ and such that each number e_{2i} lies strictly between e and $e + t/2^2$ and such that for each $x \in A_2$, there exists a point $e_x \in \{e_{2i}\}_{i=1}^{m_2 m_1 n_0}$ such that $f(x, e_x) \notin A_2$.

This process can be continued by mathematical induction. In this way we obtain sequences

$$\{m_i\}_{i=1}^{\infty}, \qquad \{e_{ij}\}_{j=1}^{m_i m_{i-1} \cdots m_2 m_1 n_0}, \qquad \{A_i\}_{i=1}^{\infty}$$

satisfying the following conditions.

(vii) Each m_i is an integer greater than 2.

(viii) All of the numbers e_{ij} are distinct and for each $i \in N$ (the natural numbers), e_{ij} lies strictly between e and $e + t/2^i$ for each $j = 1, 2, \ldots, m_i m_{i-1} \cdots m_2 m_1 n_0$.

(ix) Each set A_i is a closed subset of $[x_0, x_1]$, $A_{i+1} \subset A_i$ for each $i \in N$ and $m(A_i) > (0.9)(x_1 - x_0)$ for each $i \in N$. For each $x \in A_i$, there exists a point $e_x \in \{e_{ij}\}_{j=1}^{m_i m_{i-1} \cdots m_2 m_1 n_0}$, such that $f(x, e_x) \notin A_i$.

Let $\{e_n\}$ denote the set $\{e_{ij}: i \in N,\ 1 \le j \le m_i m_{i-1} \cdots m_2 m_1 n_0\}$ written as a strictly decreasing sequence and let $A = \bigcap_{n=1}^{\infty} A_n$.

Clearly $m(A) \ge (0.9)(x_1 - x_0) > 0$ and, by (viii) $\lim_{n\to\infty} e_n = e$. If $x \in A$, then, by (ix), $f(x, e_n) \notin A$ for infinitely many n. Finally, if $x \notin A$; since A is closed, $f(x, e) = x$ and f is continuous; it follows that there exists an $n_x \in N$ such that $f(x, e_n) \notin A$ for every n, $n \ge n_x$.

Remark 1. Theorem 1 can be extended by replacing condition (a) by the hypothesis that $f_x(x_0, e) > 0$. Furthermore an n-dimensional version of Theorem 1 is valid under appropriate conditions on $f: R^n \times R^n \to R^n$.

Our next result is an extension of part (1) of Theorem B and D.

Theorem 2. *If A is a measurable set of real numbers with $m(A) > 0$ and $f: R \times R \to R$ is a function satisfying the following conditions:*

(d) *there exists an $e \in R$ such that $f(x, e) = x$ for every $x \in R$;*

(e) *f_x and f_y exist and are continuous everywhere;*

(f) *$f_y(x, e) > 0$ for all $x \in R$;*

and (e_n) is a sequence converging to e, then, for almost all $x \in A$, $f(x, e_n) \in A$ for infinitely many n.

Proof. Let $m \in N$ and let A_m denote the set $A \cap (-m, m)$. For each $\bar{\varepsilon} > 0$, since A_m is bounded and measurable, there exists a positive integer $n = n(\bar{\varepsilon})$ and there are disjoint intervals $I_1, I_2, \ldots, I_n$ such that

$$(*) \qquad A_m = \left(\bigcup_{k=1}^{n} I_k \cup E_2 \right) \setminus E_1$$

where E_1 and E_2 each have outer Lebesgue measure less than $\bar{\varepsilon}$.

Let $\varepsilon > 0$. By $(*)$ and the conditions on f there exists a subsequence $(n_k)_{k=1}^{\infty}$ of the natural numbers such that

$$m(A_m \setminus f(A_m, e_{n_k})) < \varepsilon/2^k \quad \text{for each } k.$$

From this it follows that $\{x \in A_m : x$ is not in infinitely many of the sets $f(A_m, e_n)\}$ has measure less than ε for each $\varepsilon > 0$. Since this holds for each $m \in N$ it follows that for almost all $x \in A$, $f(x, e_n) \in A$ for infinitely many n.

Remark 2. In connection with Theorem 2, see [2].

The Baire set analogue of Theorem 2 is not true. In fact the following holds.

Theorem 3. *If* A *is a Baire subset of* R *and* $f: R \times R \to R$ *is a function satisfying* (d), (e) *and* (f) *and* (e_n) *is a sequence converging to* e, *then the set* $A \backslash \{x \in A: f(x, e_n) \in A$ *for all but finitely many* $n\}$ *is of the first Baire category.*

Proof. This result is immediate by the properties of f and the fact that A has the form $A = (G \backslash P) \cup Q$, where G is open and P and Q are sets of the first category.

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