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SLIDING HUMP TECHNIQUE AND SPACES WITH THE WILANSKY PROPERTY

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ABSTRACT. We prove that if E is a BK-AK-space whose dual E' as well is BK-AK, then $\sigma(E', F)$ and $\sigma(E', \overline{F})$ have the same convergent sequences whenever F is a subspace of E'' containing Φ and satisfying $F^{\beta} = E^{\beta}$. This extends a result due to Bennett [B$_2$] and the second author [S]. We provide new examples of BK-spaces having the Wilansky property. We show that the bidual E'' of a solid BK-AK-space E whose dual as well is BK-AK satisfies a separable version of the Wilansky property. This extends a theorem of Bennett and Kalton, who proved that l^{∞} has the separable Wilansky property.

INTRODUCTION

G. Bennett [B$_2$] and the second author [S] have independently obtained a positive answer to the following question of Wilansky: Is c_0 the only FK-space, densely containing Φ, whose β-dual is l^1? Both approaches are essentially based on a characterization of the barrelledness of certain sequence spaces by means of their β-duals. In the present paper we extend the Bennett/Stadler result, providing more examples of BK-spaces having the Wilansky property (in the sense introduced in [B$_2$]).

Let us explain the situation by considering a typical example. The classical sliding hump argument (Toeplitz/Schur) asserts that $\sigma(l^1, c_0)$-bounded sets are $\| \|_1$-bounded. The Bennett/Stadler result generalizes this to the extent that still $\sigma(l^1, E)$-bounded sets are $\| \|_1$-bounded, when $\Phi \subset E \subset c_0$ and $E^{\beta} = l^1$. The latter may be expressed equivalently by saying that every subspace E of c_0 containing Φ and having $E^{\beta} = l^1$ is barrelled. Finally, our present attempt shows that $\sigma(l^1, E)$-bounded sets are $\| \|_1$-bounded when $\Phi \subset E \subset l^{\infty}$ and $E^{\beta} = l^1$. Actually, we prove a little more. We show that $\sigma(l^1, E)$ and $\sigma(l^1, \overline{E})$ have the same convergent sequences in case $\Phi \subset E \subset l^{\infty}$ and $E^{\beta} = l^1$. This extension requires a modified technique, since both the approaches in [B$_2$] and

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[S] make use of the sectional convergence in E (when $E \subset c_0$), and the latter is no longer available (when $E \subset l^\infty$).

We obtain new classes of BK-spaces having the Wilansky property. For instance, we prove that every BK-AK-space E, such that $S(E')$ is separably complemented in E', has the Wilansky property. Here $S(E')$ denotes the space of all $y \in E'$ which have sectional convergence with respect to the norm.

We prove that the bidual E'' of a solid BK-AK-space E whose dual E' is as well BK-AK has the following separable Wilansky type property: If D is a norm dense subspace of E'' containing Φ and having $D^\beta = E^\beta$ $(= E')$, then every separable FK-space F containing D must actually contain E''. When applied to the case $E = c_0$, this provides a result of Bennett and Kalton [BK$_1$, W, p. 259].

Notation. The sections of a sequence $x \in \omega$ are denoted by

$$P_n x = \sum_{i=1}^{n} x_i e^i ,$$

where e^i are the unit vectors. If $P_n x \to x$ $(n \to \infty)$, then x is said to have sectional convergence. If E is a BK-space, then $S(E)$ denotes the space of all $x \in E$ having sectional convergence with respect to the norm on E.

Concerning all other notions from sequence space theory, we refer to the book [W].

The main theorem

This section presents our fundamental result.

Theorem 1. *Let E be a BK-AK-space such that $S(E')$ is complemented in E' with separable complement L. Let $M = S(E')^\perp$ be the annihilator of $S(E')$ in E''. Let F be any subspace of $E^{\beta\beta}$ containing Φ and suppose $F^\beta = E^\beta$ $(= E')$. Then $\sigma(E', F + M)$ and $\sigma(E', \overline{F} + M)$ have the same convergent sequences.*

Proof. We need some preparations. We may assume that E has a monotone norm (see [W, p. 104]). Let $p_1\colon E' \to S(E')$, $p_2\colon E' \to L$ be the projection operators corresponding with the decomposition $E' = S(E') \oplus L$. Notice that $E'' = S(E')^\perp \oplus L^\perp$, $L' = S(E')^\perp = M$. We define norm continuous linear operators $Q_n\colon E' \to E'$, $n \in \mathbf{N}$, by

$$Q_n y = P_n \circ p_1 y + p_2 y.$$

Then we have

$$\|y - Q_n y\| = \|p_1 y - P_n \circ p_1 y\| \to 0.$$

We have to prove that $\sigma(E', F + M)$-convergent sequences are $\sigma(E', \overline{F} + M)$-convergent. To this end, it suffices to prove that every $\sigma(E', F + M)$-null sequence is bounded in norm. Indeed, suppose this has been proved for a $\sigma(E', F+M)$-null sequence (y^n), $\|y^n\| \le K$, say. Then, for $\overline{x} \in \overline{F}$ fixed and $\varepsilon > 0$ choose $x \in F$ having $\|x - \overline{x}\| < \varepsilon/K$. Then

$$|\langle \overline{x}, y^n \rangle| \le K\|x - \overline{x}\| + |\langle x, y^n \rangle| < \varepsilon$$

for $n \ge n(\varepsilon)$.

Let (y^n) be a $\sigma(E', F+M)$-null sequence and assume it is not bounded in norm, $\|y^n\| \ge n2^n$, say. Let $v^n = y^n/n$.

I. There exist strictly increasing sequences (k_j), (n_j) of integers such that the following conditions (1) and (2) are satisfied:

(1) $\|Q_{k_{j-1}} v^{n_j}\| \le 2^{-j}$, $\quad j = 1, 2, \dots,$

(2) $\|v^{n_j} - Q_{k_j} v^{n_j}\| \le 2^{-j}$, $\quad j = 1, 2, \dots.$

Suppose $k_1, \dots, k_j$ and $n_1, \dots, n_j$ have already been defined in accordance with (1) and (2). We claim that $\|Q_{k_j} v^n\| \to 0$ $(n \to \infty)$. Since (y^n) is $\sigma(E', F+M)$-null, $(p_2 y^n)$ is bounded for $\sigma(L, M)$, hence is norm bounded, hence $\|p_2 v^n\| \to 0$. On the other hand, $y^n = p_1 y^n + p_2 y^n$ implies that $(p_1 y^n)$ is $\sigma(E', F+M)$-bounded, hence $(p_1 v^n)$ is $\sigma(E', F+M)$-null, hence is coordinatewise null in view of $\Phi \subset F$. Clearly this implies $\|P_{k_j} p_1 v^n\| \to 0$, proving our claim. But now it is clear that a choice of $n_{j+1} > n_j$ satisfying (1) is possible.

Next observe that $\|v^{n_{j+1}} - Q_k v^{n_{j+1}}\| \to 0$ $(k \to \infty)$. This permits a choice of $k_{j+1} > k_j$ in accordance with (2).

II. Let $z^j = Q_{k_j} v^{n_j} - Q_{k_{j-1}} v^{n_j} = P_{k_j} p_1 v^{n_j} - P_{k_{j-1}} p_1 v^{n_j}$, and let $\alpha_j = 1/\|z^j\|$. Then (α_j) is an l^1-sequence by (1), (2). Observe that $\alpha_j z^j \to 0$ with respect to $\sigma(E', F+M)$, but $\|\alpha_j z^j\| = 1$. Therefore, a result of Pelczyński [P] guarantees the existence of a basic subsequence $(\alpha_{j_r} z^{j_r})$ of $(\alpha_j z^j)$. To simplify the reasoning in the following, we assume that $(\alpha_j z^j)$ itself is a basic sequence in E'.

III. We claim the existence of a null sequence (λ_j) such that the sequence z, defined by

$$(*) \qquad z_k = \lambda_j \alpha_j z_k^j \quad \text{for } k_{j-1} < k \le k_j,$$

is not an element of $S(E')$.

Let G denote the subspace of E' consisting of all sequences

$$z = \sum_{j=1}^{\infty} \lambda_j \alpha_j z^j,$$

where (λ_j) is in c_0 and the series converges in norm. Define a linear operator φ: $G \to c_0$ by setting

$$\varphi(z) = \varphi\left(\sum_{j=1}^{\infty} \lambda_j \alpha_j z^j\right) = (\lambda_j).$$

φ is well defined since $(\alpha_j z^j)$ is a basic sequence by assumption. We prove that φ is continuous. Let $z \in G$, $z = \sum \lambda_j \alpha_j z^j$. Then

$$\begin{aligned} |\lambda_j| = \|\lambda_j \alpha_j z^j\| &= \left\|\sum_{i=1}^{j} \lambda_i \alpha_i z^i - \sum_{i=1}^{j-1} \lambda_i \alpha_i z^i\right\| \\ &= \left\|P_{k_j} z - P_{k_{j-1}} z\right\| \leq 2\|z\|, \end{aligned}$$

the latter in view of the monotonicity of the norm on E (and thus on E'). This proves that φ is continuous.

Let $\overline{G}$ be the norm closure of G in E'. Then φ extends to a continuous, linear operator $\overline{\varphi}$: $\overline{G} \to c_0$. Let $z \in \overline{G}$, then $z = \sum \lambda_j \alpha_j z^j$ for some sequence (λ_j), since $(\alpha_j z^j)$ is a basic sequence. But notice that $\overline{\varphi}(z) = (\lambda_j)$ by a K-space argument. So actually (λ_j) is in c_0, hence $z \in G$, proving $G = \overline{G}$.

Notice that φ is a continuous injection. This proves that φ is not surjective. For supposing it were, it would be a homeomorphism by the open mapping theorem, i.e. we would have $G \approx c_0$. But this is absurd, since no separable dual space may contain a copy of c_0. So φ is not surjective. Let (λ_j) be any null sequence which is not in the range of φ. We prove that z, defined by $(*)$, is not in $S(E')$. Indeed, $z \in S(E')$ would imply $\|z - P_{k_j} z\| \to 0$ $(j \to \infty)$. But note that

$$P_{k_j} z = \sum_{i=1}^{j} \lambda_i \alpha_i z^i,$$

hence z would be in G, which was excluded. This ends step III.

IV. We prove that $(P_k z)$ is $\sigma(E', F + M)$-convergent with limit z. Indeed, let $x \in F + M$, $k \in \mathbf{N}$, $k_{j-1} < k \leq k_j$. Then we have

$$\langle x, P_k z\rangle = \sum_{i=1}^{j-1} \lambda_i \alpha_i \langle x, z^i\rangle + \lambda_j \alpha_j \langle x, P_k z^j\rangle.$$

Here the first summand converges $(k \to \infty, k_{j-1} < k \leq k_j)$ since $\langle x, z^i\rangle \to 0$ and $(\alpha_j) \in l^1$. But the second summand converges as well in view of $\lambda_j \to 0$ $(k \to \infty, k_{j-1} < k \leq k_j)$ and

$$|\alpha_j \langle x, P_k z^j\rangle| = |\langle P_k x, \alpha_j z^j\rangle| \leq \|P_k x\| \|\alpha_j z^j\| \leq \|x\|.$$

In view of $F^\beta = E'$ this implies $z \in E'$ and so $P_k z \to z$ in $\sigma(E', F + M)$.

Now observe that the operators Q_r are $\sigma(E', F+M)$-continuous, so $Q_r(P_k z) \to Q_r z$ $(k \to \infty)$, proving $P_r z = Q_r z$, hence $z \in S(E')$. But this contradicts step III and therefore ends the proof. □

In the case where $S(E') = E'$, i.e. when E' has sectional convergence, the proof may be simplified. Here we have $M = \{0\}$, $Q_n = P_n$. This yields the following.

Corollary 1. *Let E be a BK-AK-space such that E' is as well BK-AK. Let F be a subspace of E'' containing Φ and satisfying $F^\beta = E^\beta$ $(= E')$. Then $\sigma(E', F)$ and $\sigma(E', \overline{F})$ have the same convergent sequences.* □

Spaces with the Wilansky property

An FK-space E is said to have the Wilansky property if every subspace F of E satisfying $F^\beta = E^\beta$ is barrelled in E (see [B$_2$]). In [B$_2$] and [S] it is proved that every BK-AK-space E whose dual E' is as well a BK-AK-space has the Wilansky property. Here we obtain:

Theorem 2. *Let E be a BK-AK-space such that $S(E')$ has a separable complement L in E'. Let G be any FK-space having $E \subset G \subset E^{\beta\beta}$. Then G has the Wilansky property if and only if E is of finite codimension in G.*

Proof. Necessity. Suppose E is of infinite codimension in G. Let (y^n) be a linearly independent sequence in $G \backslash E$. Since $E^\beta = G^\beta$, E is barrelled as a subspace of G, hence is closed in G. But now $F = E + \operatorname{lin}\{y^n \colon n \in \mathbf{N}\}$ is a subspace of G having $F^\beta = G^\beta$ which is not barrelled. Indeed, we may define a sequence (f_n) in G' such that f_n is 0 on $E + \operatorname{lin}\{y^1, \dots, y^{n-1}\}$ and satisfies $f_n(y^n) = n|y^n|$ (for some continuous seminorm $|\cdot|$ on G). Then $f_n \to 0$, $\sigma(G', F)$, but (f_n) is not bounded in G'.

Sufficiency. Let F be a subspace of G with $F^\beta = G^\beta$. We may assume that F contains Φ (see [B$_2$, Theorem 1]).

Let U be a barrel in F. Since $M \cap E = \{0\}$, $M = S(E')^\perp$, the space $F \cap M$ is finite dimensional. Let S be some topological complement of $F \cap M$ in M. Let B denote the unit ball in S. Note that B is $\sigma(E'', E')$-compact, since the unit ball in $M \approx L'$ is weak $*$ compact and S is of finite codimension in M. Now let $V = U + B$. Then V^0, the $\langle E'', E' \rangle$-polar of V, is $\sigma(E', F + M)$-bounded, since V spans $F + M$. By Theorem 1, $\sigma(E', F + M)$-bounded sets are norm bounded in E', so that V^0 is norm bounded in E'. Hence V^{00} is a norm neighbourhood of 0 in E'', hence $V^{00} \cap F$ is a norm neighbourhood of 0 in F, since G (and hence F) must have the topology induced by E''. We end the proof by showing $V^{00} \cap F \subset U$. By the definition of V, we have $V^{00} = \overline{U} + B$, the closure being taken in $\sigma(E'', E')$, since B is $\sigma(E'', E')$-compact. But $V^{00} \cap F = \overline{U} \cap F$ in view of $B \cap F = \{0\}$. Since F has only finitely many dimensions "outside E", we deduce that $\overline{U} \cap F = U$, which ends the proof of Theorem 2. □

Corollary 2. [B_2, § 6]. *c and cs have the Wilansky property.* □

More generally, a BK-AK-space E has the Wilansky property if $S(E')$ is of finite codimension in E', and the same is true for any G having $E \subset G \subset E^{\beta\beta}$ such that E is of finite codimension in G. In a forthcoming paper [NS], we use this fact to prove that for every invertible permanent triangular matrix A whose inverse A^{-1} is a bidiagonal matrix, the convergence domain c_A has the Wilansky property.

Remark. In Theorems 1,2, the assumption that E has separable dual may be replaced by any condition ensuring that c_0 does not embed into E'. See for instance [Kw].

Separable Wilansky property

It is clear from Theorem 2 that the bidual E'' of a BK-AK-space E whose dual E' is as well BK-AK does not have the Wilansky property unless E has finite codimension in E''. Nevertheless, the bidual space E'' satisfies some weaker Wilansky type property, which might be called the separable Wilansky property.

Theorem 3. *Let E be a solid BK-AK-space whose dual E' is as well BK-AK. Let D be a norm dense subspace of E'' containing Φ and satisfying $D^\beta = E^\beta$ $(= E')$. Then every separable FK-space F which contains D, actually contains E''.*

Proof. Let $x \in E''$ be fixed. Since D is a norm dense in E'', it is also $\tau(E'', E')$-sequentially dense in E'', i.e. there exists a sequence (x^n) in D which converges to x in $\tau(E'', E')$. We claim that $\tau(E'', E')|D = \tau(D, E')$.

Indeed, by Theorem 1, $\sigma(E', D)$ and $\sigma(E', E'')$ have the same convergent sequences, hence the same compact sets [W, p. 252]. This implies $\tau(E'', E')|D = \tau(D, E')$.

Consequently, the sequence (x^n) is Cauchy in $(D, \tau(D, E'))$. We prove that the inclusion mapping i: $(D, \tau(D, E')) \to F$ is continuous. This is a consequence of Kalton's closed graph theorem (see [BK_2, Theorem 5]), for $\sigma(E', D)$ is sequentially complete. Indeed, since $\sigma(E', D)$ and $\sigma(E', E'')$ have the same convergent sequences, they also have the same Cauchy sequences. But $\sigma(E', E'')$ is sequentially complete as a consequence of the fact that E, and hence $E' = E^\alpha$, is solid. This proves that $\sigma(E', D)$ is sequentially complete.

Since i: $(D, \tau(D, D^\beta)) \to F$ is continuous, the sequence (x^n) is Cauchy in F, and hence converges to some $\overline{x} \in F$. From K-space reasons, we have $x = \overline{x}$, proving $x \in F$. □

Certainly, in Theorem 3, the solidity of the space E may be replaced by the condition that $\sigma(E', E'')$ is sequentially complete.

Corollary 3. (*Compare* [BK_1, *Theorem 3*].) *Let F be a separable FK-space containing Φ and suppose $F \cap l^\infty$ is norm dense in l^∞. Then $l^\infty \subset F$.*

Proof. This follows from Theorem 3 and the fact that every norm dense subspace D of l^∞ satisfies $D^\beta = l^1$ (see [W, Lemma 16.3.3]). $\square$

The result of Bennett and Kalton has been generalized by Snyder [Sn] to a nonseparable version. He proves that every FK-space F containing Φ and satisfying $F + c_0 = l^\infty$ must have $F = l^\infty$.

Scarce copies

The concept of scarce copies of sequence spaces has been introduced by Bennett [B_1]. He proves that every scarce copy of ω and l^1 is barrelled, but that all other standard sequence spaces do not have this property. For instance, c_0 does not have any barrelled scarce copy at all (see [B_1] for details). Here we obtain another negative result on the barrelledness of scarce copies.

Theorem 4. *Let E be a FK-AB-space contained in l^∞ such that $E^\gamma \subset bs$. Then E does not have any barrelled scarce copy.*

Proof. Suppose $\Sigma(E, r)$ is a barrelled scarce copy of E. This implies $\Sigma(E, r)^\beta \subset E^f = E^\gamma$, the latter since E has AB (see [W, p. 167]). Therefore $\Sigma(E, r)^\beta \subset bs$.

We prove that $\Sigma(c_0, r)$ is a barrelled scarce copy of c_0, thus obtaining a contradiction, since c_0 has no barrelled scarce copies. Since c_0 has the Wilansky property, barrelledness of $\Sigma(c_0, r)$ will be a consequence of $\Sigma(c_0, r)^\beta \subset l^1$. So let $y \notin l^1$. Since $c_0^\gamma = l^1$, there exists $x \in c_0$ such that $xy \notin bs$, hence $xy \notin \Sigma(E, r)^\beta$. Let $z \in \Sigma(E, r)$ be chosen with $xyz \notin cs$. By the definition of $\Sigma(E, r)$, there exist $z^1, \dots, z^n \in \sigma(E, r)$ having $z = z^1 + \cdots + z^n$. This implies $xyz^i \notin cs$ for some i. We claim that $xz^i \in \sigma(c_0, r) \subset \Sigma(c_0, r)$. Since $z^i \in E \subset l^\infty$, we have $xz^i \in c_0$. On the other hand,

$$c_n(xz^i) \le c_n(z^i) \le r_n$$

for every n implies $xz^i \in \sigma(c_0, r)$. This proves $y \notin \Sigma(c_0, r)^\beta$. $\square$

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