SOME SPECIAL 2-GENERATOR KLEINIAN GROUPS

BERNARD MASKIT

(Communicated by Irwin Kra)

ABSTRACT. We explore the following question. Let G be the subgroup of $\mathrm{PSL}(2, \mathbb{C})$ generated by the elements A and B , where A has two fixed points, and B maps one fixed point of A onto the other; when is G discrete?

In conjunction with their work on the generalization of Jørgensen's inequality, Gehring and Martin [G-M] asked the following question (oral communication): Let $G_0 = \langle a, b \rangle$ be the subgroup of $\mathrm{PSL}(2, \mathbb{C})$ generated by an elliptic element a of order 6, and a parabolic element b , where b maps one fixed point of a onto the other; is G_0 discrete?

In this note, we give an affirmative answer to this question, describe some of the properties of this interesting group, and explore the following more general question:

Suppose a and b are elements of $\mathrm{PSL}(2, \mathbb{C})$, where a has exactly two fixed points, and b maps one of these fixed points to the other. When is $G_0 = \langle a, b \rangle$ discrete?

1. We start with a necessary condition.

Proposition 1. *If G_0 is discrete, then either a is elliptic of order 2, 3, 4, or 6, or b is elliptic of order 2.*

Proof. Normalize so that the fixed points of a are at 0 and ∞ , where $b(0) = \infty$. If b is not a half-turn (i.e., an elliptic element of order 2), then $b(\infty) \neq 0$, so a and bab^{-1} have exactly the one fixed point at ∞ in common. Then the commutator $c = [a, bab^{-1}] = a(bab^{-1})a^{-1}(bab^{-1})^{-1}$ is parabolic with fixed point at ∞ . In a discrete group, a loxodromic (including hyperbolic) element and a parabolic element cannot share a fixed point, hence a is elliptic. It follows that $\langle a, c \rangle$ is a discrete group of Euclidean motions; hence the order of a must be 2, 3, 4, or 6. $\square$

2. If b is a half-turn, then G_0 preserves the fixed point set of a , so G_0 conjugates a into a^{-1} ; hence G_0 is a $\mathbb{Z}_2$ extension of $\langle a \rangle$, the cyclic group

Received by the editors December 2, 1987 and, in revised form, April 27, 1988.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 30F40.

Research supported in part by NSF Grant #DMS 8701774.

© 1989 American Mathematical Society
0002-9939/89 \$1.00 + \$.25 per page

generated by a . It follows that, in this case, G_0 is discrete if and only if $\langle a \rangle$ is discrete. Of course $\langle a \rangle$ is discrete if and only if a is either loxodromic or elliptic of finite order.

3. From here on we assume that b is not a half-turn, and that a is a geometrically primitive elliptic transformation of order $\alpha = 2, 3, 4$, or 6 (an elliptic transformation is geometrically primitive if it is conjugate to a rotation of the form $z \rightarrow e^{2\pi i/q} z$, $q \in \mathbb{Z}$). We will also assume that b maps one fixed point of a onto the other. We will show below that $G_0 = \langle a, b \rangle$ is discrete if b is parabolic, hyperbolic, or a geometrically primitive elliptic transformation of finite order. When it is necessary to make it clear which of these groups we are referring to, we will label the group as $G_0(\alpha, \delta)$, where α is the order of a , and either $\delta = 0$, or $\delta = \infty$, or δ is an integer ≥ 3 . This has the following meaning. If $\delta = 0$, then b is hyperbolic; if $\delta = \infty$, then b is parabolic; otherwise, b is a geometrically primitive elliptic transformation of order δ .

There are two other numbers that come up in conjunction with the number α . The first is β , defined by $1/\alpha + 1/\beta = 1/2$; if $\alpha = 2$, then $\beta = \infty$. The second is $\gamma = \beta/2$. Note that β is again an integer (including ∞); also γ is an integer for $\alpha = 2, 3$, or 4, but not for $\alpha = 6$.

We will show that these groups are discrete by explicitly constructing fundamental polyhedra, and using Poincaré's polyhedron theorem. A proof of this theorem, along with the requisite definitions, can be found in [M, p. 73 ff].

We will need the following computation.

Lemma 1. *Let a be an elliptic transformation of order α , with fixed points at x and y . Let b be a Möbius transformation where $b(x) = y$. If $\alpha = 2$, then the commutator $[a^{-1}, b^{-1}]$ is parabolic, and if $\alpha = 3$ or 4, then $[a^{-1}, b^{-1}]$ is elliptic of order γ .*

Proof. Normalize so that a has fixed points at 0 and ∞ , and so that $b(0) = \infty$. Write

$$a = \begin{pmatrix} e^{\pi i/\alpha} & 0 \\ 0 & e^{-\pi i/\alpha} \end{pmatrix}, \quad b = \begin{pmatrix} s & t \\ -t^{-1} & 0 \end{pmatrix},$$

and compute the trace of the commutator. $\square$

One of the conclusions of Poincaré's polyhedron theorem is that one can read off a presentation, in the usual group theoretical sense, from the identifications of the sides of the polyhedron. One is also often interested in knowing which elements of a subgroup of $\text{PSL}(2, \mathbb{C})$ are parabolic. Since every commutative subgroup of rank > 1 in a discrete group of hyperbolic motions is purely parabolic, it suffices to know the rank 1 parabolic subgroups. If one has a given (convex) fundamental polyhedron P for the discrete group G , then every fixed point of a parabolic element in G is equivalent to a point on the boundary of P [M, p. 123]. It follows that one can also read off generators for rank 1 parabolic subgroups from the identifications of the sides of P .

A *presentation* for a discrete subgroup H of $\mathrm{PSL}(2, \mathbb{C})$ is of the form $H = \langle a_1, a_2, \dots : w_1^{\alpha_1} = w_2^{\alpha_2} \cdots = 1 \rangle$, where the a_j are generators of H , the w_j are words in these generators, and the α_j are either non-negative integers or the symbol " ∞ ". The presentation has the following meaning. If one ignores those words where $\alpha_j = 0$ or ∞ , then one has an ordinary presentation of the algebraic structure of H . For each j where $\alpha_j = \infty$, w_j is parabolic, and the maximal commutative subgroup of H containing w_j has rank 1. Further, if g is a parabolic element of H , where the maximal commutative subgroup of H containing g has rank 1, then g is conjugate to a power of some w_j with $\alpha_j = \infty$. For the purpose of uniformity of presentation, one sometimes has a word of the form w^0 in the presentation; these have no meaning and should be ignored.

Our polyhedra will all be convex, and will be constructed in the upper half-space, $\mathbf{H}^3 = \{(z, t) | z \in \mathbb{C}, t > 0\}$, which we consider to be endowed with the hyperbolic metric:

$$ds^2 = t^{-2}(|dz|^2 + dt^2).$$

A (finite) convex polyhedron P in $\mathbf{H}^3$ is the intersection of a finite number of (hyperbolic) half-spaces D_j ; the half-space D_j has boundary S_j in $\mathbf{H}^3$, and S_j has Euclidean boundary C_j in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We will describe D_j by B_j , the restriction of its Euclidean boundary to $\widehat{\mathbb{C}}$. We denote the face of D lying on the side S_j by the same letter S_j ; this should cause no confusion.

4. In this section, we explicitly construct fundamental polyhedra with identifications for $\alpha = 2, 3$, and 4 , and for $\delta = 0$, $\delta \geq 3$, and $\delta = \infty$.

We start with some definitions. A Kleinian group G is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$; it is of the first kind if every point of $\widehat{\mathbb{C}}$ is a limit point of G (that is, every point is the limit of a sequence of points of the form $\{g_m(x)\}$, where x is any point of $\mathbf{H}^3$); it is of the second kind otherwise.

If G is of the second kind, then the set of points on $\widehat{\mathbb{C}}$ at which G acts discontinuously is called the regular set, and is denoted by $\Omega = \Omega(G)$. If G is finitely generated with Ω/G connected (which is the only case that occurs here), then Ahlfors' finiteness theorem asserts that Ω/G can be represented as a closed Riemann surface of genus p , from which a finite number of points have been removed (these are identified as special points of order ∞), and the covering is branched over finitely many points (these are identified as special points of order s , where s is the order of branching). If in addition, every component of Ω is simply connected, then the signature of G is defined as $(p, n; s_1, \dots, s_n)$, where n is the total number of special points, and $s_1, \dots, s_n$ are their orders.

A function group is a finitely generated Kleinian group where some component of Ω is kept invariant by the entire group. Function groups have more complicated signatures (see [M, p. 271 ff]).

Theorem 1. For $\alpha = 2, 3$, or 4 , the group $G_0(\alpha, \delta)$ is discrete. It has the (Kleinian group) presentation:

$$G_0 = \langle a, b : a^\alpha = b^\delta = [a^{-1}, b^{-1}]^\gamma = 1 \rangle.$$

Further, $G_0(\alpha, \delta)$ is of the first kind if $(\alpha, \delta) = (4, 4)$, $(4, 3)$, or $(3, 3)$; it is of the second kind otherwise. For the groups that are of the second kind, if $\delta \neq 0$, then every component of Ω is a circular disc, $\Omega(G_0)/G_0$ is connected and has signature $(0, 3; \gamma, \delta, \delta)$. If $\delta = 0$, then $\Omega(G_0)$ is connected but not simply connected; it can be topologically described by a function group signature as follows: the signature has one part of basic signature $(0, 3; \alpha, \alpha, \gamma)$, and there is an α -connector between the two special points of order α .

Proof. We first remark that the statement about function group signatures can be restated as follows: $\Omega(G_0)/G_0$ is a single Riemann surface of signature $(1, 1; \gamma)$, and there is a nondividing simple loop on this surface so that α is the smallest positive power for which this loop lifts to a loop in Ω .

CONSTRUCTION OF $G_0(\alpha, 0)$, $\alpha = 2, 3, 4$

Normalize so that $b(z) = \lambda^2 z$, $\lambda > 1$, and so that a has its fixed points at 1 and λ^2 . If $\alpha \neq 2$, choose a , as opposed to a^{-1} , so that the center of the isometric circle of a lies in the upper half-plane. Define P by the sides $B_1 = \{z \mid |z| > 1\} \cup \{\infty\}$; $B_2 = \{z \mid |z| < \lambda\}$; B_3 is the outside of the isometric circle of a ; and B_4 is the outside of the isometric circle of a^{-1} ; if $\alpha = 2$, then $S_3 = S_4$ (see Figure 1, for $\alpha = 3$). Observe that $b(S_1) = S_2$, and $a(S_3) = S_4$.

For $\alpha \neq 2$, the faces S_3 and S_4 meet at an angle of $2\pi/\alpha$; this edge is not equivalent to any other edge. The faces S_1 and S_2 meet the faces S_3 and S_4 at an angle of π/β ; these four edges are equivalent, and form a cycle of

FIGURE 1

edges, the sum of the angles in this cycle is $2\pi/\gamma$. Lemma 1 shows that the corresponding cycle transformation, $[a^{-1}, b^{-1}]$, has order γ .

For $\alpha = 2$, the polyhedron has no edges; there is one reflection relation, namely $a^2 = 1$; and there is one infinite cycle transformation, $[a^{-1}, b^{-1}]$, which is parabolic.

The information above shows that our polyhedron with identifications satisfies the hypotheses of Poincaré's polyhedron theorem. We conclude that $G_0(\alpha, 0)$ is discrete, and has the given presentation.

Since the boundary of this fundamental polyhedron is a fundamental domain for the action of G on $\hat{\mathbb{C}}$ [M, p. 116], we see at once that $\Omega(G_0)/G_0$ is connected; hence G_0 is a function group. Identifying the sides of this fundamental domain, we see that $\Omega(G_0)/G_0$ is a torus with one special point of order γ . To obtain the Kleinian group signature, note that there is a simple loop in $\Omega(G_0)$ that is precisely invariant under $\langle a \rangle$; the projection of this loop is a simple loop, which, when raised to the α th power, lifts to a loop. It follows from the planarity theorem (see [M, p. 251]) that since we are on a torus with one boundary component, this planar regular covering is completely determined by the ramification number at the special point, and this one loop, together with the number α .

CONSTRUCTION OF $G_0(\alpha, \delta)$, $\alpha = 2, 3, 4$, $2 < \delta < \infty$

We next take up the case that b is a geometrically primitive elliptic transformation of order $\delta > 2$. We normalize so that $b(z) = e^{2\pi i/\delta} z$, and so that the fixed points of a are at $e^{\pm\pi i/\delta}$. If $\alpha \neq 2$, then we further normalize, by perhaps replacing a with a^{-1} , so that $a^{-1}(\infty)$ lies inside the unit disc.

We define the sides of P by $B_1 = \{z: -\pi/\delta < \arg(z) < -\pi/\delta + \pi\}$; $B_2 = \{z: \pi/\delta - \pi < \arg(z) < \pi/\delta\}$; C_3 is the circle passing through the fixed points of a and making an angle of π/α with the unit circle (this angle is measured outside C_3 and inside the unit circle); then a is the composition of reflection in C_3 followed by reflection in the unit circle. Let B_3 be the outside of C_3 . For $\alpha = 2$, set $C_4 = C_3$; for $\alpha > 2$, let $C_4 = a(C_3)$; in either case, define B_4 so that a maps B_3 onto the complement of B_4 . Note that C_4 also makes an angle of π/α with the unit circle. These define a polyhedron P ; observe that $b(S_1) = S_2$, and $a(S_3) = S_4$ (see Figure 2 for the case $\alpha = 3$, $\delta = 6$).

It is an easy computation to see that the origin is contained inside C_3 if $\alpha > \delta$; C_3 passes through the origin if $\alpha = \delta$; and the origin lies outside C_3 if $\alpha < \delta$. If $\alpha \geq \delta$, which can occur only when $(\alpha, \delta) = (4, 4)$, $(4, 3)$, or $(3, 3)$, the boundary of P intersects the boundary of $\mathbf{H}^3$ in a finite number of points, so the group is of the first kind; if $\alpha < \delta$, then the boundary of P intersects the boundary of $\mathbf{H}^3$ in a fundamental domain for the action of G_0 on $\hat{\mathbb{C}}$.

If $\alpha \neq 2$, then, exactly as above, the sides S_1 and S_2 meet the sides S_3 and S_4 at the angle π/β , and these four edges form a cycle of order γ . Also, the

sides S_3 and S_4 meet at an angle of π/α ; this one edge is its own cycle of order α .

If $\alpha = 2$, then P has no edges, and has one infinite cycle; as above, Lemma 1 shows that the corresponding infinite cycle transformation is parabolic.

In either case, we can conclude from Poincaré's polyhedron theorem that $G_0(\alpha, \delta)$ is discrete, and has the stated presentation. If $(\alpha, \delta) \neq (4, 4)$, $(4, 3)$, or $(3, 3)$, then the fundamental domain for G_0 at the boundary of P has two components (these are shaded in Figure 2). These two components are identified by a , so one easily sees, by folding together the sides of this fundamental domain, that $\Omega(G_0)/G_0$ has signature $(0, 3; \gamma, \delta, \delta)$. Since every uniformization of a surface with this signature is given by a Fuchsian group, every component of $\Omega(G_0)$ is a circular disc.

FIGURE 2

CONSTRUCTION OF $G_0(\alpha, \infty)$, $\alpha = 2, 3, 4$

We normalize so that $b(z) = z + 2$, so that a has its fixed points at ± 1 , and so that if $\alpha > 2$, then the center of the isometric circle of a lies in the upper half-plane. We construct the polyhedron P as follows. $B_1 = \{z \mid \operatorname{Re}(z) > -1\}$; $B_2 = \{z \mid \operatorname{Re}(z) < 1\}$; B_3 is the outside of the isometric circle of a ; and B_4 is the outside of the isometric circle of a^{-1} . These form a polyhedron P , where the sides S_1 and S_2 are identified by b , and the sides S_3 and S_4 are identified by a (See Figure 3 for the case $\alpha = 3$).

As in the two cases above, if $\alpha \neq 2$, the sides S_1 and S_2 meet the sides S_3 and S_4 in four edges, each with an angle of π/β ; these four edges form a cycle

FIGURE 3

of order γ . There is also an edge between S_3 and S_4 , where the cycle consists of only this one edge, and has order α . If $\alpha = 2$, then a^2 is a reflection relation; P has no edges, and has one infinite cycle; the infinite cycle relation is: $[a^{-1}, b^{-1}]^\infty = 1$.

The fundamental domain on the boundary of P on the sphere at infinity has two components that are connected via a . Hence, $\Omega(G_0)/G_0$ is connected. Folding together the sides of this fundamental domain, we see that $\Omega(G_0)/G_0$ has signature $(0, 3; \gamma, \infty, \infty)$. As above, this implies that every component of $\Omega(G_0)$ is a circular disc. $\square$

5. The above construction does not work for $\alpha = 6$, for in this case the sides S_1 and S_2 meet the sides S_3 and S_4 at an angle of $\pi/3$, so that the sum of the angles in the cycle of these edges is $4\pi/3$. We get around this difficulty by constructing a larger group generated by reflections. We start by constructing a hyperbolic polyhedron with four sides, where the sides meet at certain specified angles. If S_m and S_n are sides of the polyhedron P , then we denote the interior angle between S_m and S_n by θ_{mn} .

Theorem 2. *Let a be elliptic of order $\alpha = 2, 3, 4$, or 6 , and let b be hyperbolic, parabolic, or a geometrically primitive elliptic transformation of finite order $\delta > 2$, where b maps one fixed point of a onto the other. Then there is a polyhedron P , bounded by four sides, $S_1, \dots, S_4$, where $\theta_{12} = \pi/\delta$ (if $\delta = 0$, then S_1 and S_2 do not intersect, even on the sphere at infinity; if $\delta = \infty$, then S_1*

and S_2 are tangent at the sphere at infinity); $\theta_{13} = \pi/2$; $\theta_{14} = \pi/\beta$ (these sides are tangent at the sphere at infinity if $\alpha = 2$); $\theta_{23} = \pi/2$; $\theta_{24} = \pi/2$; $\theta_{34} = \pi/\alpha$. Denote reflection in S_j by r_j . Then the following hold.

- (i) $a = r_3 r_4$, and $b = r_1 r_2$.
- (ii) The group $\tilde{G}(\alpha, \delta)$, generated by $r_1, \dots, r_4$ is discrete and has the presentation $\langle r_1, \dots, r_4: r_1^2 = \dots = r_4^2 = (r_1 r_2)^\delta = (r_1 r_3)^\beta = (r_1 r_4)^\beta = (r_2 r_3)^2 = (r_2 r_4)^2 = (r_3 r_4)^\alpha = 1 \rangle$.
- (iii) If $\alpha \geq \delta$, then $\tilde{G}(\alpha, \delta)$ is of the first kind; it is of the second kind otherwise. For $\delta \neq 0$, the action of $\tilde{G}(\alpha, \delta)$ on Ω is that of the $(2, \beta, \delta)$ -triangle group, generated by r_1, r_2 , and r_4 . For $\delta = 0$, the fundamental domain for the action of $\tilde{G}(\alpha, 0)$ on its regular set is a quadrilateral with angles $\pi/2, \pi/2, \pi/2$, and π/β .

Proof. We start with the observation, again using Poincaré's polyhedron theorem, that for conclusions (i) and (ii), it suffices to construct the polyhedron P , with the sides meeting at the correct angles, where a and b are as described in conclusion (i).

CONSTRUCTION OF $\tilde{G}(\alpha, 0)$.

Normalize a and b as above so that $b(z) = \lambda^2 z$, $\lambda > 1$, and so that a has its fixed points at 1 and λ^2 ; we can also assume that if $\alpha > 2$, then the center of the isometric circle of a lies in the upper half-plane. Let $B_1 = \{z | |z| < \lambda^2\}$; $B_2 = \{z | |z| > \lambda\} \cup \{\infty\}$; B_3 is the upper half-plane; and B_4 is the outside of the isometric circle of a (the fundamental domain for conclusion (iii) is shaded in Figure 4).

FIGURE 4

FIGURE 5

CONSTRUCTION OF $\tilde{G}(\alpha, \delta)$, $2 < \delta < \infty$

Normalize so that $b(z) = e^{2\pi i/\delta} z$; a has its fixed points at $e^{\pm\pi i/\delta}$; and if $\alpha > 2$, the center of the isometric circle of a lies inside the unit circle. Let $B_1 = \{z | \pi/\delta - \pi < \arg(z) < \pi/\delta\}$; B_2 is the upper half-plane; B_3 is the unit disc; and B_4 is the outside of the circle passing through the fixed points of a , and making an angle of π/α with the (inside of the) unit circle (see Figure 5 for the case that $\alpha = 3$, and $\delta = 6$; the fundamental domain for conclusion (iii) is shaded).

CONSTRUCTION OF $\tilde{G}(\alpha, \infty)$

Normalize so that $b(z) = z + 2$, so that a has its fixed points at ± 1 , and so that if $\alpha > 2$, then the center of the isometric circle of a lies in the upper half-plane. Let $B_1 = \{z | \operatorname{Re}(z) < 1\}$; B_2 is the right half-plane; B_3 is the upper half-plane; and B_4 is the outside of the isometric circle of a (the fundamental domain for this group is shaded in Figure 6). $\square$

6. Let $G(\alpha, \delta)$ be the orientation preserving half of $\tilde{G}(\alpha, \delta)$. Then $G(\alpha, \delta)$ is generated by $a = r_3 r_4$, $b = r_1 r_2$, and $c = r_1 r_4$, and has the presentation $G(\alpha, \delta) = \langle a, b, c : a^\alpha = b^\delta = c^\beta = (ac^{-1})^2 = (bc^{-1})^2 = (a^{-1}b^{-1}c)^2 = 1 \rangle$.

Using the explicit constructions of the fundamental polyhedra for G_0 and $\tilde{G}$, it is easy to see that for $\alpha = 2, 3$, or 4 , $G_0(\alpha, \delta)$ is a subgroup of index 2 in $G(\alpha, \delta)$. We can make this explicit by observing that, up to the boundaries, we obtain a fundamental polyhedron for $G(\alpha, \delta)$ from that of $\tilde{G}(\alpha, \delta)$ by reflection in S_2 , and then we obtain the given fundamental polyhedron for $G_0(\alpha, \delta)$ by reflecting that fundamental polyhedron in S_3 . We have shown the following.

FIGURE 6

Proposition 2. For $\alpha = 2, 3$, and 4 , and for all possible δ ,

$$[G(\alpha, \delta): G_0(\alpha, \delta)] = 2.$$

Proposition 3. For every possible δ , $G(6, \delta) = G_0(6, \delta)$.

Proof. We first prove this result for $\delta = \infty$. We renormalize so that $a(z) = e^{-\pi i/3} z$, so that $b(z)$ has its fixed point at 1, and so that $b(0) = \infty$. Then $b(z) = (2z - 1)/z$. With this new normalization, we have $C_1 = \{z | \operatorname{Re}(z) = 1\}$, $C_2 = \{z | |z| = 1\}$, $C_3 = \{z | \operatorname{Im}(z) = 0\}$, and $C_4 = \{z | \arg(z) = \pi/6\}$ (see Figure 7). It is easy to compute reflections in these lines. Writing these reflections as matrices in $\operatorname{PGL}(2, \mathbb{C})$, we obtain the following.

$$\begin{aligned} r_1 &= \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, & r_2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ r_3 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & r_4 &= \begin{pmatrix} e^{i\pi/6} & 0 \\ 0 & e^{-i\pi/6} \end{pmatrix}, \end{aligned}$$

from which we compute

$$\begin{aligned} a &= \begin{pmatrix} e^{-i\pi/6} & 0 \\ 0 & e^{i\pi/6} \end{pmatrix}, & b &= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \\ c &= \begin{pmatrix} -e^{-i\pi/6} & 2e^{i\pi/6} \\ 0 & e^{i\pi/6} \end{pmatrix}. \end{aligned}$$

Now set $e = [a, bab^{-1}]$, so that $e(z) = z - 2$. Then a straightforward computation shows that $e^{-1}aea = c^{-1}$. Hence $G(6, \delta) = G_0(6, \delta)$.

FIGURE 7

There is an obvious homomorphism, given by conclusion (ii) of Theorem 2, from $G(6, \infty)$ onto $G(6, \delta)$ (this is an isomorphism for $\delta = 0$), from which we conclude that $G(6, \delta)$ is generated by a and b for every δ . $\square$

7. We conclude with some remarks.

(i) We first note that $G(6, 0)$ is a function group representing a surface of signature $(0, 4; 2, 2, 2, 3)$, so the fact that this group has only two generators is related to the observation that the Fuchsian group of signature $(0, 4; 2, 2, 2, 3)$ can be generated by two hyperbolic elements [P-R].

(ii) We note that for $\delta \neq 0$, the group $\tilde{G}(\alpha, \delta)$ contains the following four triangle groups, obtained by taking the defining reflections three at a time: the $(2, 2, \delta)$ -triangle group, which is finite for δ finite, and Euclidean for $\delta = \infty$; the $(2, \beta, \delta)$ -triangle group, which is finite if $1/\beta + 1/\delta > 1/2$, Euclidean if $1/\beta + 1/\delta = 1/2$, and Fuchsian otherwise; the Euclidean $(2, \alpha, \beta)$ -triangle group; and the finite $(2, 2, \alpha)$ -triangle group. Of course, $G(\alpha, \delta)$ contains the orientation preserving half of these groups.

(iii) Since every $\tilde{G}(\alpha, \delta)$ with $\delta \neq 0$ contains the Euclidean $(2, \alpha, \beta)$ -triangle group, none of the groups we consider here has a compact fundamental polyhedron in $\mathbf{H}^3$.

(iv) The above analysis omits two cases. One expects that, in general, if a and b are as above, where b is elliptic of finite order, but not geometrically primitive, then $\langle a, b \rangle$ is not discrete. It is not clear if there are any exceptions.

(v) If a and b are as above, and b is hyperbolic, then any quasiconformal deformation of $\langle a, b \rangle$ is again discrete, and in general b is loxodromic. There are also limits of such groups that are discrete. It is not known if there are any other such discrete groups, where b is loxodromic.

REFERENCES

- [G-M] F. W. Gehring and G. Martin, (to appear).
- [M] B. Maskit, *Kleinian groups*. Springer-Verlag, 1988.
- [P-R] N. Purzitsky and G. Rosenberger, *Correction to "Two generator Fuchsian groups of genus one"*, Math. Z. **132** (1973), 261–262.

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT STONY BROOK, STONY BROOK, NEW YORK 11794-3651