SUBGROUPS OF ALMOST SEPARABLE TORSIONFREE ABELIAN GROUPS

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ABSTRACT. A characterization is given of those subgroups of an almost separable torsionfree abelian group which inherit separability from the group.

INTRODUCTION

In this paper we give a fairly general answer to the question “Which pure subgroups of a separable torsionfree abelian group G are separable?”. By investigating the splitting properties typical of separability, it was proved classically by Fuchs [FII] that direct summands of G are separable, and recently, by L. G. Nongxa in [N], that “strongly pure” subgroups of G are separable.

An attempt to deal with the “necessity” side of the question shows that it is too loosely phrased: it is easy to provide examples of completely decomposable pure subgroups of a separable group G , having little or nothing to do with what we experience as the “separation structure” of G . We thus use our previous investigation of the t -local properties typical of separability [M1, M3], to specify the initial question by demanding that the separable subgroups “inherit” their separability from G .

To make our request more specific, recall that, for an element g of the separable group G , the “separation type” $T_G(g)$ of g denotes (what amounts to) the isomorphism class $[X]$ of a minimal rank completely decomposable summand X of G containing g . We say that the separable pure subgroup B of G “preserves separation type” if, for each $b \in B$, $T_G(b) = T_B(b)$. Note that direct summands, as well as strongly pure subgroups, preserve separation type. The question we consider then becomes: “Which pure subgroups of G are separable with preserved separation type?”. To answer it, we deal with the more general class of “almost separable groups” introduced in [M3], and prove that a subgroup B of an almost separable group G is pure, almost separable,

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with preserved separation type, if and only if B is a TT^* -subgroup of G (TT^* -subgroups, defined in [M3], are subgroups which derive their t -local structure from G). An easy reduction then shows that TT^* -subgroups are the answer in the separable case as well (Theorem 3.2).

As a final by-product, we note that, while being quite different in general, in separable groups TT^* -subgroups coincide with strongly pure subgroups: so we get the converse of Nongxa's result.

0. NOTATION AND PRELIMINARY RESULTS

"Group" means "torsionfree abelian group". Notation and terminology are those of [FII, M1, M3]: we recall here some of it. $B \leq_* G$ means that B is a pure subgroup of G ; $B \leq_{\oplus} G$, that B is a direct summand of G . If $G = A \oplus B$, we will write $g = a \oplus b$ for the induced direct decomposition of $g \in G$, with no further explanation of the symbols. In the proper context, $n \mid g$ ($n \nmid g$) means: n divides (does not divide) g . For $G = \langle a \rangle_* \oplus H$, we say $g \in G$ is substitutable to a in the given decomposition of G , if $G = \langle g \rangle_* \oplus H$; for $g = a \oplus h$, this holds if and only if $\chi(h) \geq \chi(a)$ [M3, Lemma 0]. $g \in G$ is proper with respect to $B \leq_* G$, if $\langle g, B \rangle_* = \langle g \rangle_* \oplus B$; g is extractible in G if $\langle g \rangle_* \leq_{\oplus} G$, in which case the type $t(g)$ of g is called an extractible type of G ; $T(G)$ denotes the set of extractible types of G .

$\mathbb{T}$ will be the set of all types; for $t', t'' \in \mathbb{T}$, $t'' >_{\text{n.c.}} t'$ means t'' is greater than or noncomparable to t' ; and analogously for $\geq_{\text{n.c.}}$. For $t \in \mathbb{T}$, we consider the functorial pure subgroups of G : $G(t) = \{g \in G \mid t(g) \geq t\}$ and $G^*\{t\} = \bigcap_{U \in \mathcal{U}_t} U$, where $\mathcal{U}_t$ is the family of all pure corank one subgroups of G whose cotype is $\leq t$; $G^*\{t\}$ is then the reject of R in G , where R is a type t subgroup of $\mathbb{Q}$.

We complete the definitions given in [M3] by

Definition 0.1. Let E be a subset of the group G . We define $\chi_G(E)$, the characteristic of E in G , as follows: if $E \neq \emptyset$, $\chi_G(E) = \sup_{g \in E} \chi_G(g)$; while $\chi_G(\emptyset) = 0$, thus adding 0, as an absolute minimum, to the set of characteristics. We also add ∞ as an absolute maximum, and define $\chi_G(0) = \infty$. Note that for $g \in G$ we have $\chi_G(g) = \chi_G(\{g\})$. We say $e \in E$ is a faithful element of E if $\chi_G(e) = \chi_G(E)$.

As a consequence, $\chi_G(E) = \infty$ whenever $0 \in E$; in particular, for any subgroup E of G . This definition enables us to deal with cosets $g + H$ of the subgroup H of G : e.g., if $H \leq_* G$, then $\chi_G(g + H) = \chi_{G/H}(g + H)$, thus simplifying notation; g is proper with respect to H if g is a faithful element of $g + H$.

The newly defined characteristic respects inclusion between arguments as well as between indices, and in fact satisfies the same properties as does the usual characteristic of elements:

Lemma 0.2. If E, E' are subset of G , then:

- (a) if $E \subseteq E'$, then $\chi_G(E) \leq \chi_G(E')$;

- (b) $\chi_G(E + E') \geq \inf(\chi_G(E), \chi_G(E'))$;
 (c) If $\langle E, E' \rangle_* \equiv \langle E \rangle_* \oplus \langle E' \rangle_* \pmod{(E \cap E')_*}$, then
 $\chi_G(E + E') = \inf(\chi_G(E), \chi_G(E'))$.

Remark. Definition 0.1 allows a natural definition for a “nice subgroup” of the torsionfree group G : $H \leq G$ is nice if each coset of H in G contains a faithful element.

We recall now from [M3] the definition and principal properties of T^* - and TT^* -subgroups. We will use these results in the following without explicit reference.

A subgroup B of G is a T^* -subgroup: $B \leq_{T^*} G$, if $\forall t \in \mathbb{T}$, B satisfies

- (a₁): $B^*\{t\} = B \cap G^*\{t\}$, and
 (a₂): $(B + G^*\{t\})_* = B + G^*\{t\}$.

B is a TT^* -subgroup of G : $B \leq_{TT^*} G$, if B is T^* , and, $\forall t \in \mathbb{T}$, B satisfies

- (b₁): $B \cap (G(t) + G^*\{t\}) = B(t) + B^*\{t\}$, and
 (b₂): $(B + G(t) + G^*\{t\})_* = B + (G(t) + G^*\{t\})_*$ [M3, Definition 1.2].

An element $g \in G$ is a T^* element of G if g is proper with respect to $G^*\{t(g)\}$, i.e. if $\langle g, G^*\{t(g)\} \rangle_* = \langle g \rangle_* \oplus G^*\{t(g)\}$; or, equivalently, if $\langle g \rangle_*$ is a T^* -subgroup of G [M3, Definition 2.6, Proposition 2.7].

T^* -subgroups are pure [Observation 1.a: note that the proof given in [M3] is incorrect; the result follows from the fact that, for $q = t(\mathbb{Q})$, $G^*\{q\} = 0$]; direct summands are TT^* [M3, Proposition 1.4], T^* and TT^* are inductive and transitive.

We use the handiness of Definition 0.1 to sharpen the result of Proposition 1.6b in [M3].

Proposition 0.3. *Let $A \leq B \leq G$. Then $A \leq_{T^*} G$ implies $A \leq_{T^*} B$, and $A \leq_{TT^*} G$ implies $A \leq_{TT^*} B$.*

Proof. We have $A^*\{t\} \leq A \cap B^*\{t\} \leq A \cap G^*\{t\} = A^*\{t\}$, thus $A \leq B$ satisfies (a₁). To prove $(A + B^*\{t\})_* \leq_* B$, note that, if $a \in A$, $a + A^*\{t\} \subseteq a + B^*\{t\} \subseteq a + G^*\{t\}$, therefore $\chi_A(a + A^*\{t\}) \leq \chi_B(a + A^*\{t\}) \leq \chi_B(a + B^*\{t\}) \leq \chi_G(a + B^*\{t\}) \leq \chi_G(a + G^*\{t\})$, which by (a₂) equals $\chi_{A+G^*\{t\}}(a + G^*\{t\}) = \chi_A(a + A \cap G^*\{t\}) = \chi_A(a + A^*\{t\})$ by (a₁). Hence $\chi_B(a + B^*\{t\}) = \chi_B(a + A^*\{t\}) = \chi_B(a + A \cap B^*\{t\}) = \chi_{A+B^*\{t\}}(a + B^*\{t\})$, and $(A + B^*\{t\})/B^*\{t\} \leq_* B/B^*\{t\}$, as desired. The proof for TT^* runs analogously. $\square$

A group G is t -bihomogeneous if $G = G(t)$ and $G^*\{t\} = 0$. If the group G is t -bihomogeneous, its pure subgroups are t -bihomogeneous and TT^* [M3, Lemma 2.2, Corollary 2.4]. A t -bihomogeneous subgroup is TT^* iff it satisfies (a₁), (a₂) for the type t [M3, Lemma 2.3]. We restate explicitly

Lemma 2.8 [M3]. *For a subgroup $X = \bigoplus_{t \in \mathbb{T}} X_t$ of G , where each X_t is t -bihomogeneous, the following are equivalent: (i) $X \leq_{TT^*} G$; (ii) $X \leq_{T^*} G$; (iii) $\forall t \in \mathbb{T}$, $X_t \leq_{T^*} G$; (iv) $\forall t \in \mathbb{T}$, $\forall x \in X_t$, x is a T^* element of G .*

Finally, we say an element $g \in G$ is separable in G if $g \in X \leq_{\oplus} G$, with X completely decomposable; g is almost separable in G if $g \in X \leq_T G$, with X completely decomposable; a group G is almost separable if every $g \in G$ is almost separable in G . We will use

Definition 0.4. For $g \in G$ and $t \in \mathbb{T}$, introduce:

- the lower t -characteristic of g in G : $\chi'_{t,G}(g) = \chi_G(g + G^*\{t\})$, and
- the upper t -characteristic of g in G : $\chi''_{t,G}(g) = \chi_G(g + (G(t) + G^*\{t\})_*)$;
- the separation type of g in G is the set $T_G(g) = \{t \in \mathbb{T} \mid \chi'_{t,G}(g) < \chi''_{t,G}(g)\}$ (as in [M3]; note that $\leq$ always holds). If $B \leq G$, we say the inclusion of B in G preserves separation types if, $\forall b \in B$, $T_B(b) = T_G(b)$.

The characterization of the separation type used in the introduction is in

Lemma 3.3 [M3]. *If g is almost separable in G , then $T_G(g) = T(X)$, where X is any minimal rank completely decomposable T^* -subgroup of G containing g .*

Note that in this case X is short, i.e. $|T(X)| = \text{rk}(X)$.

Lower and upper t -characteristics increase along homomorphisms:

Proposition 0.5. *Let $G, \bar{G}$ be (torsionfree abelian) groups, and $f: G \rightarrow \bar{G}$ a homomorphism. Then, $\forall g \in G$ and $\forall t \in \mathbb{T}$, we have*

$$\chi'_{t,G}(g) \leq \chi'_{t,\bar{G}}(f(g)), \quad \text{and} \quad \chi''_{t,G}(g) \leq \chi''_{t,\bar{G}}(f(g)).$$

Proof. First, let f be surjective, and w.l.o.g. $\bar{G} = G/K$, where $K = \ker f$. From [M1, Proposition 1.2.c] we get $(K + G^*\{t\})/K \leq (G/K)^*\{t\}$, thus, writing $G^*\{K, t\}$ for the preimage, in G , of $(G/K)^*\{t\}$, and, using Lemma 0.2 above, we have $\chi'_{t,G}(g) = \chi_G(g + G^*\{t\}) \leq \chi_G(g + K + G^*\{t\}) \leq \chi_G(g + G^*\{K, t\}) = \chi_{G/K}(g + K + (G/K)^*\{t\}) = \chi'_{t,\bar{G}}(f(g))$. The proof for χ'' runs analogously, since

$$\frac{K + (G(t) + G^*\{t\})_*}{K} \leq \left(\frac{K + G(t)}{K} + \frac{K + G^*\{t\}}{K} \right)_* \leq \left(\frac{G}{K}(t) + \frac{G^*}{K}\{t\} \right)_*,$$

therefore

$$\begin{aligned} \chi''_{t,G}(g) &= \chi_G(g + (G(t) + G^*\{t\})_*) \leq \chi_G(g + K + (G(t) + G^*\{t\})_*) \\ &\leq \chi_{G/K}(g + K + ((G/K)(t) + (G/K)^*\{t\})_*) = \chi''_{t,\bar{G}}(f(g)). \end{aligned}$$

We complete the proof if we show that, whenever $h \in H \leq G$, $\chi'_{t,H}(h) \leq \chi'_{t,G}(h)$, $\chi''_{t,H}(h) \leq \chi''_{t,G}(h)$. From [M1, Proposition 1.2.b] we have $H^*\{t\} \leq H \cap G^*\{t\}$, thus $\chi'_{t,H}(h) = \chi_H(h + H^*\{t\}) \leq \chi_H(h + H \cap G^*\{t\}) = \chi_{H+G^*\{t\}}(h + G^*\{t\}) \leq \chi_G(h + G^*\{t\}) = \chi'_{t,G}(h)$. The proof for χ'' runs analogously, since $(H(t) + H^*\{t\})_* \leq (G(t) + G^*\{t\})_*$. $\square$

1. PRESERVING SEPARATION TYPES

We prove now that, in looking for subgroups B of an almost separable group G which are almost separable with preserved separation type (see Definition 0.4), we may restrict ourselves to T^* -subgroups of G ; and similarly for the separable case.

Proposition 1.1. *Let G be almost separable, B an almost separable pure subgroup of G . Then the following are equivalent:*

- (i) *The inclusion of B in G preserves separation types (i.e. $\forall b \in B$, $T_B(b) = T_G(b)$);*
- (ii) *$B \leq_{T^*} G$.*

Proof. (i) $\Rightarrow$ (ii) will be proved by contradiction. Suppose B is not T^* in G because (a₁) does not hold, i.e. there is $b \in B \cap G^*\{t\} \setminus B^*\{t\}$. Then $\chi'_{t,G}(b) = \infty$, thus for each type $t' \leq t$, $t' \notin T_G(b)$; since B is almost separable, $b \in Y \leq_{T^*} B$, Y completely decomposable; $b \notin B^*\{t\}$, thus $b \notin Y^*\{t\}$, hence $T_Y(b) \ni t' \leq t$, thus $T_Y(b) = T_B(b) \neq T_G(b)$, against (ii).

Assume next (a₁) holds, but (a₂) fails. Then there is $b \in B \setminus B^*\{t\}$ and $n \in \mathbb{N}$ such that $n \mid (b + G^*\{t\})$ in $G/G^*\{t\}$ but not in $(B + G^*\{t\})/G^*\{t\} \cong B/B^*\{t\}$; thus $\chi'_{t,G}(b) > \chi'_{t,B}(b)$. Let now B be almost separable, $b = b' \oplus b^*$, with $b^* \in B^*\{t\}$, and $b' \in Y'$, a completely decomposable T^* -subgroup of B such that $Y' \cap B^*\{t\} = 0$ (so $T_B(b')$ consists of types $\leq t$). Then $n \mid (b + G^*\{t\}) = b' + G^*\{t\}$ in $G/G^*\{t\}$, but $n \nmid (b + B^*\{t\}) = b' + B^*\{t\}$ in $B/B^*\{t\}$, thus $n \mid b'$ (in B , hence in G). If G is almost separable, write analogously $b' = x' \oplus x^*$, where $x^* \in G^*\{t\}$, and $x' \in X'$, a completely decomposable T^* -subgroup of G with $X' \cap G^*\{t\} = 0$. We have $n \nmid b'$, but $n \mid (b' + G^*\{t\}) = x' + G^*\{t\}$, thus $n \mid x'$. Hence $x^* \neq 0$, and $T_G(b')$ contains types $>_{n.c.} t$, thus is different from $T_B(b')$, in contradiction to (i).

(ii) $\Rightarrow$ (i) follows easily from the more general observation that, if B and G are any groups with $B \leq_{T^*} G$, and if b is almost separable in B , then, for Y completely decomposable such that $b \in Y \leq_{T^*} B \leq_{T^*} G$, we have $T_B(b) = T_Y(b) = T_G(b)$, as required. $\square$

Similar conclusions hold for separable groups, in view of the next lemma.

Lemma 1.2. (i) *Let b be an almost separable element of $B \leq_{T^*} G$, where G is separable. Then b is separable in B . Thus*

- (ii) *Almost separable T^* -subgroups of separable groups are separable.*

Proof. Let $b \in Y \leq_{T^*} B \leq_{T^*} G$, Y finite rank completely decomposable. Then $Y \leq_{T^*} G$, thus $Y \leq_{TT^*} G$, hence by [M3, Proposition 2.9] $Y \leq_{\oplus} G$, thus $Y \leq_{\oplus} B$. $\square$

Corollary 1.3. *Let G be separable, B a separable pure subgroup G . Then the following are equivalent:*

- (i) *The inclusion of B in G preserves separation types;*
- (ii) *$B \leq_{T^*} G$.*

2. TT^* -SUBGROUPS OF ALMOST SEPARABLE GROUPS

We investigate now the T^* -subgroups of almost separable groups G , to discover which ones are almost separable.

Lemma 2.1. *Let G be almost separable. Then $\forall t, \tau \in \mathbb{T}$, $G(t) \cap G^*\{\tau\} = G(t)^*\{\tau\}$.*

Proof. One inclusion is always true. Let then $g \in G(t) \cap G^*\{\tau\}$: we have $g \in X \leq_{T^*} G$, and we may choose $T(X)$ to consist of types $\geq t$, $>_{\text{n.c.}} \tau$. Any $U \leq_* G(t)$ of corank 1 and cotype $\leq \tau$ will then contain X , hence g : thus, $g \in G(t)^*\{\tau\}$. $\square$

Proposition 2.2. *Let G be almost separable, $B \leq_{T^*} G$. The following are equivalent:*

- (i) B is almost separable;
- (ii) $\forall t \in \mathbb{T}$, B satisfies $(b_t): B \cap (G(t) + G^*\{t\}) = B(t) + B^*\{t\}$.

Proof. (i) $\Rightarrow$ (ii). Let B be almost separable, and $b \in B \cap (G(t) + G^*\{t\})$. If $b \in G^*\{t\}$, we are done by (a_1) . If not, then any minimal rank completely decomposable $X \leq_{T^*} G$ containing b is contained in $(G(t) + G^*\{t\}) \setminus G^*\{t\}$: thus $T_G(b) = T(X) = \{t\} \cup T'$, where the types of T' are $>_{\text{n.c.}} t$. Since B is almost separable, we can choose Y completely decomposable of minimal rank such that $b \in Y \leq_{T^*} B$, thus $T_B(b) = T_Y(b) = T(Y)$. Y is a T^* -subgroup of G as well, hence $T_Y(b) = T_G(b) = \{t\} \cup T'$. Then $Y = \langle y \rangle_* \oplus Y^*$, with $Y^* \leq B^*\{t\}$, and $b = y \oplus y^* \in B(t) + B^*\{t\}$, as desired.

To prove (ii) $\Rightarrow$ (i), let $b \in B$, $b \in X$, a short completely decomposable T^* -subgroup of G ; $|T_G(b)| = |T(X)| = n$. We show b is almost separable in B , by induction on n . If $n = 1$, by Proposition 0.3 b is a T^* element of B , hence almost separable in B . Let $n > 1$ and t minimal in $T(X)$; write $b = x \oplus x^* \in \langle x \rangle_* \oplus G^*\{t\}$. It will be enough to prove that x is substitutable, in this decomposition, by a T^* element $b' \in B$: for then, $b = b' \oplus y^*$ with $y^* \in G^*\{t\}$ yields $y^* \in B^*\{t\}$, and $|T_G(y^*)| = |T_G(b) \setminus \{t\}| = n - 1$; so, by induction hypothesis, y^* is almost separable in B , and since the separation types of y^* and of b' are disjoint, $b = b' + y^*$ is almost separable in B [M3, Lemma 2.8].

Since $b \in B \cap (G(t) + G^*\{t\}) = B(t) + B^*\{t\}$, we can write $b = b_t + b^*$, $b_t \notin G^*\{t\}$, so that $b_t - x = x^* - b^* = g^* \in G(t) \cap G^*\{t\}$; we have then $b_t = x + g^* \in H = B \cap (\langle x \rangle_* \oplus (G(t) \cap G^*\{t\})) \leq_* B \cap G(t) = B(t)$. Now consider

$$\frac{H}{B \cap G(t) \cap G^*\{t\}} \simeq \frac{H + (G(t) \cap G^*\{t\})}{G(t) \cap G^*\{t\}} \leq \frac{\langle x \rangle_* \oplus (G(t) \cap G^*\{t\})}{G(t) \cap G^*\{t\}} \simeq \langle x \rangle_*.$$

The left-hand quotient has rank ≤ 1 , type $\leq t$; but $b_t \in H \setminus B \cap G(t) \cap G^*\{t\}$; thus, $B \cap G(t) \cap G^*\{t\}$ is a corank 1, cotype t pure subgroup of $H = H(t)$,

hence it splits: $H = \langle b' \rangle_* \oplus B \cap G(t) \cap G^*\{t\}$. Now b' is a type t T^* element of $B(t)$, since $H \leq_* G$ and $B \cap G(t) \cap G^*\{t\} = B(t) \cap B^*\{t\} \geq B(t)^*\{t\}$; moreover, $b_t = b' \oplus b^{**}$, with $b^{**} \in B(t) \cap B^*\{t\}$; thus $b' = x \oplus (g^* - b^{**}) \in x + G^*\{t\}$, hence $\chi_G(b') \leq \chi_G(x)$.

If we show now that $\chi_G(b') \geq \chi_G(x)$, we prove that b' is substitutable to x in $\langle x \rangle_* \oplus G^*\{t\}$, and at the same time that b' is a T^* element of G : for then $t(b') = t(x) = t$, and $\langle b', G^*\{t\} \rangle_*$ (which is equal to $\langle x \rangle_* \oplus G^*\{t\}$) will equal $\langle b' \rangle_* \oplus G^*\{t\}$.

We have $x = b' + (b^{**} - g^*)$, with $b' \in G(t)$, $b^{**} - g^* \in G(t) \cap G^*\{t\} = G(t)^*\{t\}$ by Lemma 2.1; we get our inequality if we prove that the decomposition of x is direct; and we will get this by showing that b' is a T^* element of $G(t)$. We know that $\langle b' \rangle_* \oplus B \cap G(t) \cap G^*\{t\}$ is pure in $B(t)$, thus, via the natural isomorphism, $\langle b' \rangle_* \oplus G(t) \cap G^*\{t\} \leq_* B(t) + G(t) \cap G^*\{t\}$. We prove that this last group is pure in $G(t)$, by showing that it is equal to $G(t) \cap (B + G^*\{t\})$. One inclusion is obvious. Let then $g_t = b'' + g^* \in G(t) \cap (B + G^*\{t\})$: we have $b'' = g_t - g^* \in B \cap (G(t) + G^*\{t\}) = B(t) + B^*\{t\}$, thus $b'' = b_{tt} + b^*$, $b^* + g^* = g_t - b_{tt} \in G(t) \cap G^*\{t\}$; therefore, $g_t = b_{tt} + (b^* + g^*) \in B(t) + G(t) \cap G^*\{t\}$. This shows $\langle b' \rangle_* \oplus G(t)^*\{t\} \leq_* G(t)$, as desired. $\square$

We complete our result by

Theorem 2.3. *Let G be almost separable. For $B \leq_{T^*} G$, the following are equivalent:*

- (i) $B \leq_{TT^*} G$;
- (ii) B satisfies (b_1) ;
- (iii) B is almost separable.

Proof. (i) $\Rightarrow$ (ii) holds by definition, while (ii) $\Rightarrow$ (iii) follows from Proposition 2.2. To prove (iii) $\Rightarrow$ (i), we must show B satisfies (b_1) , (b_2) for each type t .

(b_1) follows from Proposition 2.2.

(b_2) To show $H = B + (G(t) + G^*\{t\})_* \leq_* G$, let $n \mid h = b + g \in H$, where $b \in B$, $g \in (G(t) + G^*\{t\})_*$. As before, w.l.o.g. $b \in Y'$, a completely decomposable T^* -subgroup of B , with $T(Y')$ consisting of types $< t$. For t' maximal in $T_G(b)$, we have $Y' \cap G^*\{t'\} = 0$, $G^*\{t'\} \geq (G(t) + G^*\{t\})_* \ni g$, $Y' \oplus G^*\{t'\} \leq_* G$; thus $h = b \oplus g$ is direct, hence $b = nb'$ with $b'' \in Y' \leq B$, $g = ng'$ with $g' \in (G(t) + G^*\{t\})_*$, as desired. $\square$

Corollary 2.4. *TT^* -subgroups of separable groups are separable. In fact, let G be separable: for $B \leq_{T^*} G$, the following are equivalent:*

- (i) $B \leq_{TT^*} G$;
- (ii) B satisfies (b_1) ;
- (iii) B is separable.

Proof. Since separable groups are almost separable, we only need to show (ii) $\Rightarrow$ (iii). This follows from Theorem 2.3 and Lemma 1.2. $\square$

3. CONCLUSIONS

The results of Paragraphs 1 and 2 can now be summed up to answer our initial question: “Which pure subgroups of a separable group G inherit separability from G ?”

Theorem 3.1. (i) *If G is almost separable, the pure almost separable subgroups of G with preserved separation type are exactly the TT^* -subgroups of G .*

(ii) *If G is separable, the pure separable subgroups of G with preserved separation type are exactly the TT^* -subgroups of G .*

Proof. (i) follows from Theorem 2.3, Proposition 1.1; (ii) from Corollary 2.4, Corollary 1.3. $\square$

In [N], L. G. Nongxa proved that strongly pure subgroups of separable groups are separable. Theorem 3.1 allows us to characterize the strongly pure subgroups of separable groups, via the following observation:

Proposition 3.2. *A subgroup of a separable group is strongly pure if and only if it is a TT^* -subgroup.*

Proof. From the proof of Theorem 4 in [N], it is easy to derive that strongly pure subgroups of separable groups (which are proved there to be separable) preserve separation type: thus, by our last theorem, they are TT^* -subgroups. On the other hand, a TT^* -subgroup B of a separable group G is strongly pure: for then, by Corollary 2.4, B is separable; by the proof of Lemma 1.2 (i), if Y is a finite rank completely decomposable summand of B containing b , $Y \leq_{\oplus} G$; and since any projection $\pi_Y: G \rightarrow Y$ yields a morphism of G in B sending b onto itself, B is strongly pure. $\square$

We thus get the converse of Nongxa’s result:

Corollary 3.3. *A subgroup of a separable group is strongly pure if and only if it is pure, separable, with preserved separation type.*

Let us finally note that, if the group G is not separable, the two notions of “strongly pure subgroup” and “ TT^* -subgroup” of G do not coincide. In particular, there are easy examples of TT^* -subgroups of almost separable groups, which are not strongly pure: for instance, in a bihomogeneous group any finite rank pure subgroup is TT^* , while only those which are direct summands are strongly pure.

REFERENCES

- [FII] L. Fuchs, *Infinite Abelian Groups II*, Academic Press, London-New York, 1973.
- [M1] C. Metelli, *On type-related properties of torsionfree abelian groups*, Abelian Group Theory, Proc. Honolulu Conference 1982–1983, Lecture Notes in Math., vol. 1006, Springer, Berlin, 1984, pp. 253–267.

- [M3] —, *Almost separable torsionfree abelian groups*, Abelian Group Theory, Proc. Oberwolfach Conference, 1985, Gordon and Breach, 1987, pp. 275–289.
- [N] L. G. Nongxa, *Strongly pure subgroups of separable torsion-free abelian groups*, Trans. Amer. Math. Soc. **290** (1985), 363–373.

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