AN APPLICATION OF HADAMARD-LÉVY'S THEOREM TO A SCALAR INITIAL VALUE PROBLEM

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ABSTRACT. A global inversion theorem due to Hadamard and Lévy is applied to a Volterra integral operator acting in a space of continuous functions to obtain a one-sided criterion for global existence and uniqueness of a scalar initial value problem.

The aim of the present paper is to establish an existence and uniqueness theorem for a scalar initial value problem, by using a global inversion theorem due to Hadamard and Lévy.

We obtain a one-sided condition which improves a special case of a previous result established in [8].

The following theorem is known as the Hadamard-Lévy theorem.

Theorem 1. *Let E, F be two Banach spaces and $f: E \rightarrow F$ be a C^1 map for which*

- (1) $f'(x) \in \text{Isom}(E, F)$ for every $x \in E$. If there exists a continuous map $\omega: \mathbf{R}_+ \rightarrow \mathbf{R}_+^*$ such that
- (2) $\int_0^\infty \frac{ds}{\omega(s)} = +\infty$,
- (3) $\|[f'(x)]^{-1}\| \leq \omega(\|x\|)$, $x \in E$,

then f is a C^1 global diffeomorphism. Here by a C^1 global diffeomorphism we understand a one-to-one map which is C^1 , onto and whose inverse is also a C^1 map. By $\text{Isom}(E, F)$ we denote the set of all linear continuous isomorphisms of E onto F .

One can easily see that for some positive constants a, b the maps $\omega_1(t) = 1$, $\omega_2(t) = at + b$, $\omega_3(t) = (at + b) \ln(t + 2)$, $\omega_4(t) = (at + b) \ln(t + 2) \ln \ln(t + 3)$, $t \geq 0$ satisfy condition (2).

A proof of Theorem 1 may be found in [2, 6–9, 11]. Here we have presented a somewhat more general statement cf. [5]. In [1, 3, 4, 8–11], the Hadamard-Lévy

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theorem was used to obtain existence and uniqueness theorems for nonlinear boundary value problems.

Throughout the paper J will denote the closed bounded interval $[a, b]$ and F will denote the Banach space of all real continuous functions defined on J , endowed with the usual sup-norm.

Lemma 2. *Let $v \in F$, $v_1 = \frac{1}{2}(v + |v|) = \max(v, 0)$ and $c = \exp(\sup_{t \in J} \int_a^t v(s) ds)$. Consider the linear bounded operator $T: F \rightarrow F$ defined as*

$$(4) \quad (Tk)(t) = k(t) + \int_a^t \exp\left(\int_s^t v(\tau) d\tau\right) v(s)k(s) ds \quad k \in F, t \in J.$$

Then the following assertions hold

- (i) $c \leq \|T\| \leq 2 \exp(\int_a^b v_1(s) ds)$;
- (ii) $\|T\| = c$ provided that $v \geq 0$;
- (iii) the map $h = Tk$ is the unique solution in F of the integral equation

$$(5) \quad h(t) - \int_a^t v(s)h(s) ds = k(t) \quad t \in J.$$

Proof. Denote $w(s, t) = \exp(\int_s^t v(\tau) d\tau)$, $w_1(s, t) = \exp(\int_s^t v_1(\tau) d\tau)$, $s, t \in J$. One can easily see that

$$(6) \quad \int_a^t w(s, t)v(s) ds = w(a, t) - 1 \quad t \in J$$

$$(7) \quad \int_a^t w_1(s, t)v_1(s) ds = w_1(a, t) - 1 \quad t \in J$$

$$(8) \quad w(s, t) \leq w_1(s, t) \quad a \leq s \leq t \leq b.$$

Let $t \in J$ and $k \in F$. Then by (6), (7) and (8) we obtain

$$\begin{aligned} |(Tk)(t)| &\leq \|k\| \left(1 + \int_a^t w(s, t)|v(s)| ds\right) \\ &= \|k\| \left(1 + \int_a^t w(s, t)(2v_1(s) - v(s)) ds\right) \\ &= \|k\| \left(1 + 2 \int_a^t w(s, t)v_1(s) ds - \int_a^t w(s, t)v(s) ds\right) \\ &\leq \|k\| \left(2 - w(a, t) + 2 \int_a^t w_1(s, t)v_1(s) ds\right) \\ &= \|k\|(2w_1(a, t) - w(a, t)) \leq 2\|k\|w_1(a, b). \end{aligned}$$

Thus the right inequality in assertion (i) is proved. Let $k_0 \in \mathbf{R}$ and $k(t) = k_0$ for every $t \in J$. Then

$$(Tk)(t) = k_0 \left(1 + \int_a^t w(s, t)v(s) ds\right) = k_0 w(a, t) \quad t \in J$$

whence $\|Tk\| = c\|k\|$. Thus the proof of the left inequality in (i) is complete. Assertion (ii) follows at once from (i) and from the inequality

$$|(Tk)(t)| \leq \|k\| \left(1 + \int_a^t w(s, t) v(s) ds \right) \quad t \in J, k \in F.$$

By Banach's contraction principle (cf. [8, Lemma 3.1]), equation (5) has a unique solution. Integration by parts shows that the map $h = Tk$ verifies (5).

It would be interesting to know whether assertion (ii) holds without the assumption $v \geq 0$.

Theorem 3. *Let $f: J \times \mathbf{R} \rightarrow \mathbf{R}$ be a continuous map for which its partial derivative with respect to the second argument, denoted by f'_x , exists and is continuous. Suppose that there exist an integrable map $g: J \rightarrow \mathbf{R}_+$ and a continuous increasing map $\omega: \mathbf{R}_+ \rightarrow (1, \infty)$ for which the following conditions hold*

$$(9) \quad \int_0^\infty \frac{ds}{\omega(s)} = +\infty$$

$$(10) \quad f'_x(t, x) \leq g(t) \ln \omega(|x|) \quad t \in J, x \in \mathbf{R}.$$

The for every $x_0 \in \mathbf{R}$ the initial value problem

$$(11) \quad \dot{x} = f(t, x), \quad x(a) = x_0$$

has a unique global solution $x: J \rightarrow \mathbf{R}$.

Proof. Let $x_0 \in \mathbf{R}$ and define $V: F \rightarrow F$ as

$$(Vx)(t) = x(t) - x_0 - \int_a^t f(s, x(s)) ds, \quad t \in J, x \in F.$$

Note that V is of C^1 class,

$$V'(x)(h)(t) = h(t) - \int_a^t f'_x(s, x(s)) h(s) ds, \quad t \in J, x, h \in F$$

and $V'(x) \in \text{Isom}(F, F)$ for every $x \in F$.

Consider the equation $V'(x)(h) = k$ where $x, k \in F$ are supposed to be known. Then by Lemma 2 the unique solution of the above equation is

$$h(t) = k(t) + \int_a^t \exp \left(\int_s^t f'_x(\tau, x(\tau)) d\tau \right) \cdot f'_x(s, x(s)) k(s) ds, \quad t \in J,$$

and

$$\|[V'(x)]^{-1}(k)\| = \|h\| \leq 2 \exp \left(\int_a^b \max(f'_x(s, x(s)), 0) ds \right).$$

By (10) we obtain

$$(12) \quad \|[V'(x)]^{-1}\| \leq 2 \exp \left(\int_a^b g(s) \ln \omega(\|x\|) ds \right) \quad x \in F.$$

If $\int_a^b g(s) ds \leq 1$, then it follows from (11) that

$$\|[V'(x)]^{-1}\| \leq 2\omega(\|x\|), \quad x \in F.$$

Then Hadamard-Lévy theorem implies that V is a C^1 global diffeomorphism, whence the equation $Vx = 0$ has a unique solution $x \in F$, which is the unique solution of (11).

Next, consider the case $\int_a^b g(s) ds > 1$. Let $a = t_0 < t_1 < t_2 < \dots < t_n = b$ be a division of J such that

$$(13) \quad \int_{t_i}^{t_{i+1}} g(s) ds \leq 1, \quad i = 0, 1, \dots, n-1.$$

Note that

$$(14) \quad f'_x(t, x) \leq g(t) \ln \omega(\|x\|), \quad t \in [t_i, t_{i+1}], x \in \mathbf{R},$$

$$i \in \{0, 1, \dots, n-1\}.$$

By (13) and (14) there exists a unique solution $\varphi_0: [t_0, t_1] \rightarrow \mathbf{R}$ for the initial value problem

$$(IVP_0) \quad \dot{x} = f(t, x), \quad x(t_0) = x_0.$$

Denote $x_1 = \varphi_0(t_1)$. By (13) and (14) there exists a unique solution $\varphi_1: [t_1, t_2] \rightarrow \mathbf{R}$ for the initial value problem

$$(IVP_1) \quad \dot{x} = f(t, x), \quad x(t_1) = x_1.$$

Construct inductively the solutions $\varphi_i: [t_i, t_{i+1}] \rightarrow \mathbf{R}$, $i = 1, 2, \dots, n-1$, of the initial value problems

$$(IVP_i) \quad \dot{x} = f(t, x), \quad x(t_i) = x_i,$$

where $x_i = \varphi_{i-1}(t_i)$.

One can easily see that the map $\varphi(t) = \varphi_i(t)$, $t \in [t_i, t_{i+1}]$, $i \in \{0, 1, \dots, n-1\}$ is of C^1 class, verifies the equations $\dot{\varphi}(t) = f(t, \varphi(t))$, $t \in J$, $\varphi(a) = x_0$, and is the unique map possessing these properties.

The above theorem improves a special case (namely $E = \mathbf{R}$) of the following theorem:

Theorem 4[8, Theorem 3.3]. *Let E be a Banach space and $f: [0, a] \times E \rightarrow E$ a continuous map for which its partial derivative with respect to the second argument, denoted by f'_x , exists and is continuous.*

Suppose that there exist a constant $k > 0$ and a continuous increasing map $\omega: \mathbf{R}_+ \rightarrow [1, \infty)$ satisfying condition (9) and a constant $k > 0$ such that

$$\|f'_x(t, x)\| \leq k \ln \omega(\|x\|), \quad t \in [0, a], x \in E.$$

Then for every $x_0 \in E$ the initial value problem

$$\dot{x} = f(t, x), \quad x(0) = x_0$$

has a unique global solution $x: [0, a] \rightarrow E$.

Remarks. One can easily see that there is a class of functions $f: J \times \mathbf{R} \rightarrow \mathbf{R}$ for which the growth condition (10) on f'_x holds but the Wintner condition [12]:

$$(14) \quad |f(t, x)| \leq g(t)\omega(|x|), \quad t \in J, \quad x \in \mathbf{R}$$

where g is integrable and ω verifies (9), does not hold. See for example rapidly decreasing maps such as $f_1(t, x) = -e^x$ or $f_2(t, x) = -x^n|x|$, $t \in J$, $x \in \mathbf{R}$ where n is a positive odd number.

Note that there are also functions $f: J \times \mathbf{R} \rightarrow \mathbf{R}$ such as $f(t, x) = \sin x^2$, $t \in J$, $x \in \mathbf{R}$ for which Wintner condition (14) holds but which does not verify the growth condition (10) with ω subject to (9).

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