EXISTENCE OF MULTIPLE PERIODIC SOLUTIONS FOR A SEMILINEAR EVOLUTION EQUATION

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ABSTRACT. In this paper, we consider the existence of multiple periodic solutions for the problem

$$\begin{aligned}\frac{du}{dt} + Lu &= g(u) + h, & t > 0, \\ u(0) &= u(T),\end{aligned}$$

where L is a uniformly strongly elliptic operator with domain $D(L) = H_0^m(\Omega)$, $g: R \rightarrow R$ is a continuous mapping, $T > 0$ and $h: (0, T) \rightarrow H_0^m(\Omega)$ is a measurable function.

1. INTRODUCTION

Let $\Omega \subset R^n$ be a bounded domain with a smooth boundary $\partial\Omega$ and

$$L := \sum_{0 \leq |\alpha|, |\beta| \leq m} (-1)^{|\beta|} D^\beta (a_{\alpha\beta}(x) D^\alpha)$$

a differential operator of order $2m$ with domain $D(L) = H_0^m(\Omega)$ satisfying the following conditions.

(L1) L is uniformly strongly elliptic, that is, there exists $C > 0$ such that

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(\chi) \xi^\alpha \xi^\beta \geq C |\xi|^{2m} \quad \text{for all } \xi \in R^n \text{ and } x \in \Omega.$$

(L2) $a_{\alpha\beta} = a_{\beta\alpha}$ are real valued functions in $C^\infty(\Omega)$.

Our purpose in the present paper is to consider the existence of multiple solutions for the problem of the form

$$\begin{aligned}(\text{P}) \quad \frac{du}{dt} + Lu &= g(u) + h(t), & t > 0, \\ u(0) &= u(T),\end{aligned}$$

where $T > 0$, $h: (0, \infty) \rightarrow L^2(\Omega)$ is a T -periodic function and, $g: R \rightarrow R$ is a continuous function with $g(0) = 0$.

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The existence of periodic solutions for problems of this kind has been studied by many authors (see Nieto [8] which also contains many references). But few seem to be known about the multiplicity of the periodic solutions. In [3], Amann established a multiplicity result for the problem (P) for the case in which L is second order. The method employed in [1] is based on the super-subsolution method. Our argument in the present paper is based on the Leray-Schauder fixed point theorem.

To state our result, we need some notations. Throughout the rest of this paper, we assume that L satisfies the condition (L1) and (L2). Let $\lambda_1 < \lambda_2 < \dots < \lambda_k < \dots$ denote the eigenvalues of the selfadjoint realization in $L^2(\Omega)$ of L . If X is a Banach space, we denote by $L^p(0, T; X)$ ($1 \leq p \leq \infty$) the space of all X -valued strongly measurable functions such that $\int_0^T \|u(t)\|^p dt < \infty$. $W^{1,p}(0, T; X)$ denotes the space of functions $u \in L^p(0, T; X)$ such that $du/dt \in L^p(0, T; X)$, where du/dt is the derivative in the sense of distribution (see [4]). We denote by $|\cdot|$ the norm of $L^2(\Omega)$.

We suppose that g satisfies the following condition

$$(G) \quad -\infty < a_* \leq a^* < \lambda_1 < b_* \leq b^* < \lambda_2,$$

where

$$a_* = \inf_{s \neq t} \frac{g(t) - g(s)}{t - s}, \quad a^* = \limsup_{|t| \rightarrow \infty} \frac{g(t)}{t}, \quad b_* = \liminf_{|t| \rightarrow \infty} \frac{g(t)}{t},$$

$$\text{and } b^* = \sup_{s \neq t} \frac{g(t) - g(s)}{t - s}.$$

Theorem. Suppose that g satisfies the condition (G). Then there exists $M > 0$ such that for each T -periodic function $h \in W^{1,\infty}(0, T; L^2(\Omega))$ with

$$\sup\{|h(t)| : t \in (0, T)\} \leq M,$$

the problem (P) possesses at least two solutions in $W^{1,\infty}(0, T; L^2(\Omega))$.

Remark. In (G), we have implicitly supposed λ_1 is single.

2. PRELIMINARIES

In the following, we write H , V and V^* instead of $L^2(\Omega)$, $H_0^m(\Omega)$ and $H^{-m}(\Omega)$, respectively. The pairing between $H_0^m(\Omega)$ and $H^{-m}(\Omega)$ is denoted by $\langle \cdot, \cdot \rangle$. If $u, v \in L^2(\Omega)$, then $\langle u, v \rangle$ is the ordinary inner product of $L^2(\Omega)$. Here we observe that we may assume without any loss of generality that $\lambda_1 > 0$. In fact, the problem (P) is equivalent to $du/dt + L_1 u = g_1(u) + h$, $u(0) = u(T)$, where $L_1 u = Lu + \lambda u$ and $g_1(u) = \lambda u + g(u)$. If $\lambda + \lambda_1 > 0$, the first eigenvalue of L_1 is positive, and g_1 satisfies the condition (G) with λ_1, λ_2 replaced by $\lambda + \lambda_1, \lambda + \lambda_2$, respectively. Also we may assume that $0 < a_*$. We denote by $\|\cdot\|$ the norm of $H_0^m(\Omega)$ defined by $\|u\|^2 = \langle Lu, u \rangle$ for $u \in H_0^m(\Omega)$. From the observation above, the norm $\|\cdot\|$ is equivalent to the Sobolev norm on $H_0^m(\Omega)$. The norm of $H^{-m}(\Omega)$ is denoted by $\|\cdot\|_*$. Let V_1

and V_2 be the subspaces of $H_0^m(\Omega)$ spanned by the eigenspaces corresponding to the eigenvalues $\{\lambda_2, \lambda_3, \dots\}$ and $\{\lambda_1\}$, respectively. Then V_1 and V_2 are orthogonal in $L^2(\Omega)$. Let ϕ be a normalized eigenfunction corresponding to λ_1 . Then $\phi \in L^\infty(\Omega)$ and $V_2 = \{k\phi : k \in R\}$. We denote by P_1 and P_2 the projections from L^2 onto V_1 and V_2 , respectively. We denote by $|\Omega|$ the measure of Ω . Suppose that g satisfies the condition (G). Then there exist positive constants α , β , ρ , and δ such that

$$(2.1) \quad \alpha < \lambda_1, \quad g(t)/t \leq \alpha \quad \text{for all } t \text{ with } |t| \geq \delta.$$

and

$$(2.2) \quad \beta > \lambda_1, \quad g(t)/t \geq \beta \quad \text{for all } t \text{ with } 0 < |t| \leq \rho.$$

Lemma 1. *There exists $\omega_1 > 0$ such that*

$$\langle Lu - g(u), u \rangle \geq \omega_1 \|u\|^2 - \delta \cdot \max\{g(t) : |t| \leq \delta\} |\Omega|$$

for all $u \in V$.

Proof. From (2.1), we have that

$$\begin{aligned} \langle Lu - g(u), u \rangle &\geq \|u\|^2 - \alpha \|u\|^2 - \int_{|u| < \delta} a(u)u \, dx \\ &\geq \|u\|^2 - \alpha \|u\|^2 - \delta \cdot \max\{g(t) : |t| \leq \delta\} |\Omega| \end{aligned}$$

for each $u \in V$. Then since $\alpha < \lambda_1$, the inequality (2.3) follows:

Lemma 2. *There exist $c_0 > 0$ and $\omega_2 > 0$ such that*

$$(2.4) \quad \langle Lu - g(u), u - 2P_2u \rangle \geq \omega_2 \|u\|^2 \quad \text{for all } u \in V \text{ with } \|P_2u\| \leq c_0.$$

Proof. Let b be a positive constant with $b^* < b < \lambda_2$. We choose $d > 0$ so small that

$$(2.5) \quad (b - b^*)p^2 - (b - a_*)d^2 > 0 \quad \text{for all } p \text{ with } p \geq \rho - d.$$

We next choose $c_0 > 0$ such that $\sup_{x \in \Omega} |w(x)| \leq d$ for all $w \in V_2$ with $\|w\| \leq c_0$. In fact, we can choose such $c_0 > 0$ because $\phi \in L^\infty(\Omega)$. Let $u \in V$ such that $\|P_2u\| \leq c$. For simplicity, we put $v = P_1u$ and $w = P_2u$. Also we set $A = \{x \in \Omega : |u(x)| > \rho\}$. Then we have

$$(2.6) \quad \langle Lu - g(u), u - 2P_2u \rangle \geq \|v\|^2 - \lambda_1 \|w\|^2 - \int_{\Omega} g(u)(v - w) \, dx.$$

Let $x \in A$. If $|w(x)| > |v(x)|$, then noting that $u(x)(v - w)(x) < 0$ we find by (G) that

$$\begin{aligned} -g(u(x))(v - w)(x) &\geq a_* |w(x)|^2 - a_* |v(x)|^2 \\ &\geq a_* |w(x)|^2 - b^* |v(x)|^2. \end{aligned}$$

If $|w(x)| \leq |v(x)|$, then we have again by (G) that

$$-g(u(x))(v - w)(x) \geq a_* |w(x)|^2 - b^* |v(x)|^2.$$

Then from (2.5) and the inequalities above, we have that

$$\begin{aligned} -\lambda_1|w(x)|^2 - g(u(x))(v-w)(x) &\geq (b-\lambda_1)|w(x)|^2 - b|v(x)|^2 \\ &\quad + \{(b-b^*)|v(x)|^2 - (b-a_*)|w(x)|^2\} \\ &\geq (b-\lambda_1)|w(x)|^2 - b|v(x)|^2. \end{aligned}$$

Let $x \in \Omega \setminus A$. Then we have by (2.2) and (G)

$$(2.8) \quad -g(u(x))(v-w)(x) \geq \beta|w(x)|^2 - b^*|v(x)|^2.$$

Then we find that

$$(2.9) \quad -\lambda_1|w(x)|^2 - g(u(x))(v-w)(x) \geq (\beta-\lambda_1)|w(x)|^2 - b^*|v(x)|^2.$$

Then combining (2.7), (2.9) with (2.6), we obtain that

$$\begin{aligned} (2.10) \quad \langle Lu - g(u), u - 2P_2u \rangle &\geq \|v\|^2 - \lambda_1|w|^2 - \int_{\Omega} g(u)(v-w) dx, \\ &\geq \|v\|^2 - b|v|^2 + (\beta-\lambda_1)|w|^2. \end{aligned}$$

Since $b < \lambda_2$ and $(\beta-\lambda_1) > 0$, we can choose $\omega_2 > 0$ such that

$$\|v\|^2 - b|v|^2 + (\beta-\lambda_1)|w|^2 \geq \omega_2\|u\|^2$$

and this completes the proof.

We now define a functional $\Phi: V \rightarrow R$ by

$$(2.11) \quad \Phi(v) = \frac{1}{2}\langle Lv, v \rangle - \int_{\Omega} \int_0^{v(x)} g(\tau) d\tau dx \quad \text{for each } v \in V.$$

It is easy to see that $\langle \Phi'(v), w \rangle = \langle Lv - g(v), w \rangle$ for $v, w \in V$. It is also easy to verify that for each $w \in V_2$, the mapping $\Phi(\cdot + w): V_1 \rightarrow R$ is strictly convex. In fact, we have from (G) that

$$\begin{aligned} (2.12) \quad \langle \Phi'(v_1 + w) - \Phi'(v_2 + w), v_1 - v_2 \rangle &\geq \|v_1 - v_2\|^2 - b^*|v_1 - v_2|^2 \\ &\geq \omega_2\|v_1 - v_2\|^2 \end{aligned}$$

for each $w \in V_2$ and each $v_1, v_2 \in V_1$. This implies that $\Phi(\cdot + w)$ is strictly convex. Then we have that for each $w \in V_1$, there exists a unique element $v_0 \in V_1$ such that

$$\Phi(v_0 + w) = \min\{\Phi(v + w): v \in V_1\}.$$

Here we set $\theta(w) = v_0$. Then $\theta: V_2 \rightarrow V_1$ is a continuous mapping satisfying that

$$(2.13) \quad \langle L(\theta(w) + w) - g(\theta(w) + w), z \rangle = 0 \quad \text{for all } z \in V_1.$$

Let $w \in V_2$. Then it also follows from the strict convexity of $\Phi(\cdot + w)$ that

$$\begin{aligned} (*) \quad v &= \theta(w) \text{ if and only if} \\ \langle L(v + w) - g(v + w), z \rangle &= 0 \quad \text{for all } z \in V_1. \end{aligned}$$

We next define a continuous function $F: R \rightarrow R$ by

$$(2.14) \quad F(s) = \Phi(\theta(s\phi) + s\phi) \quad \text{for } s \in R.$$

Then it is not difficult to see that

$$(2.15) \quad F'(x) = \langle L(\theta(s\phi) + s\phi) - g(\theta(s\phi) + s\phi), \phi \rangle,$$

(See Theorem 2.3 of [2].) It also follows from (2.13) and (2.15) that if $F(s) = \min\{F(\tau): \tau \in R\}$, then

$$(2.16) \quad L(\theta(s\phi) + s\phi) - g(\theta(s\phi) + s\phi) = 0.$$

Lemma 3. (1) *There exists $r > 0$ such that $F(s) \geq 0$ for all s with $|s| \geq r$;*

(2) $\min\{F(s): s \in (0, r)\} < 0$ and $\min\{F(s): s \in (-r, 0)\} < 0$.

Proof. We have by (2.13) and Lemma 1 that

$$\begin{aligned} F'(s)s &= \langle L(\theta(s\phi) + s\phi) - g(\theta(s\phi) + s\phi), s\phi \rangle \\ &= \langle L(\theta(s\phi) + s\phi) - g(\theta(s\phi) + s\phi), \theta(s\phi) + s\phi \rangle \\ &\geq \omega_1 \|\theta(s\phi) + s\phi\|^2 - \delta \cdot \max\{g(\tau): |\tau| \leq \delta\} |s\phi| \\ &\geq s^2 \omega_1 \|\phi\|^2 - \delta \cdot \max\{g(\tau): |\tau| \leq \delta\} |s\phi|. \end{aligned}$$

Then from the inequality above, we can choose $r > 0$ such that $F(s) \geq 0$ for all s with $|s| \geq r$. We next show that (2) holds. From (2.13) and Lemma 2, we have that

$$F'(s)s = \langle L(\theta(s\phi) + s\phi) - g(\theta(s\phi) + s\phi), s\phi \rangle < 0$$

for all s with $0 < |s| \leq c_0$. Then since $F(0) = 0$, (2) follows from (1) and the inequality above.

3. PROOF OF THEOREM

The Proof of Theorem will be accomplished by a series of lemmas. From Lemma 3, we can see that there exist constants $\alpha_{\pm}, \beta_{\pm}, \alpha_{\pm}^*$, and $\beta_{\pm}^*$ such that

$$\begin{aligned} \alpha_-^* &< \alpha_- \leq \beta_- < \beta_-^* < 0 < \alpha_+^* < \alpha_+ \leq \beta_+ < \beta_+^*, \\ (3.1) \quad F(s) &= m_+ = \min\{F(\tau): \tau \in (0, r)\} \quad \text{on } (\alpha_+, \beta_+), \\ F(s) &= m_- = \min\{F(\tau): \tau \in (-r, 0)\} \quad \text{on } (\alpha_-, \beta_-) \text{ and} \\ |F'(s)| &> 0 \quad \text{for all } s \in (\alpha_{\pm}^*, \beta_{\pm}^*) \setminus (\alpha_{\pm}, \beta_{\pm}). \end{aligned}$$

For simplicity, we will assume that $m_- = m_+ = m$. Here we choose a negative number c such that $m < c < \min\{F(\alpha_{\pm}^*), F(\beta_{\pm}^*)\}$. Then from the definition above, $V_{\pm} = \{\tau \in R: F(\tau) \leq c\} \cap (\alpha_{\pm}^*, \beta_{\pm}^*)$ are closed intervals which contain (α_+, β_+) and (α_-, β_-) , respectively. We set

$$K_{\pm} = \{v \in V: \Phi(v) \leq c\} \cap \{v \in V: P_2 v \in V_{\pm}\} \quad \text{and} \quad \tilde{K}_{\pm} = \overline{\text{co}} K_{\pm},$$

where $\overline{\text{co}} A$ denotes the closed convex hull of a subset A of V . Then from (2.12), we can see that $K_{\pm}$ are bounded closed subsets of V . Since V is compactly embedded in H , $\tilde{K}_{\pm}$ is compact in H . We also have by the definition that $\tilde{K}_{\pm} \subset \{v \in V: P_2 v \in V_{\pm}\}$.

Lemma 4. *There exist mappings $Q_{\pm}: \tilde{K}_{\pm} \rightarrow K_{\pm}$ such that $Q_{\pm}$ are continuous with respect to the norm of H and $Q_{\pm}x = x$ for each $x \in K_{\pm}$.*

Proof. Let $w \in V_+$. Then since the functional $\Phi(\cdot + w)$ is convex, we have that $P_2^{-1}(w) \cap K_+$ is compact convex in H . Here we denote by Q_w the metric projection from H onto $P_2^{-1}(w) \cap K_+$. Then Q_w is continuous with respect to the norm of H . We define a mapping Q_+ from $\tilde{K}_+$ onto K_+ by $Q_+x = Q_{P_2x}x$ for each $x \in \tilde{K}_+$. Then it is easy to verify that the mapping Q_+ is continuous on $\tilde{K}_+$ and that $Q_+x = x$ for $x \in K_+$. By the same argument, we can find a continuous mapping Q_- satisfying the required property.

We next choose open sets $U_{\pm} \subset V$ such that

$$\{\theta(t\phi) + t\phi : t \in (\alpha_{\pm}, \beta_{\pm})\} \subset U_{\pm} \subset K_{\pm} \quad \text{and} \quad \sup\{\Phi(v) : v \in U_{\pm}\} < c.$$

Then we have the following lemma:

Lemma 5. *There exist $d_{\pm} > 0$ such that*

$$(3.2) \quad \|Lv - g(v)\|_* \geq d_{\pm} \quad \text{for all } v \in K_{\pm} \setminus U_{\pm}.$$

Proof. Suppose that there exists a sequence $\{v_n\} \subset K_+ \setminus U_+$ such that $\|Lv_n - g(v_n)\|_* \rightarrow 0$, as $n \rightarrow \infty$. Since K_+ is a bounded subset of V , we may assume that $v_n \rightarrow v \in K_+$ weakly in V . Then from the compactness of K_+ in L^2 , we have that $v_n \rightarrow v$ strongly in L^2 . Thus we find that $Lv - g(v) = 0$. Then from (*), it follows that $v = \theta(\|P_2v\|\phi) + \|P_2v\|\phi$. Here we suppose that $\|P_2v\| \in (\alpha_+^*, \beta_+^*) \setminus (\alpha_+, \beta_+)$. Then we have by (3.1) that

$$0 = |\langle Lv - g(v), P_2v \rangle| = |F'(\|P_2v\|)| > 0.$$

This is a contradiction. Therefore we have that $\|P_2v\| \in (\alpha_+, \beta_+)$ and $v \in U_+$. Then from (2.12) and (2.16), we find that

$$(3.3) \quad \|Lw - g(w)\|_* \|w - v\| \geq \langle Lw - g(w), w - v \rangle \geq \omega_2 \|w - v\|^2$$

for all $w \in V$ with $P_2w = P_2v$. Noting that $P_2v_n \rightarrow P_2v$, we see that

$$(3.4) \quad \lim \|Lw_n - g(w_n)\|_* = \lim \|Lv_n - g(v_n)\|_* = 0,$$

where $w_n = v_n - P_2v_n + P_2v$ for each $n \geq 1$. Then combining (3.4) with (3.3), we obtain that v_n converges to v strongly in V . Since $K_+ \setminus U_+$ is strongly closed in V , we find that $v \in K_+ \setminus U_+$. This is a contradiction. Thus we have shown that there exists $d_+ > 0$ such that (3.2) holds for all $v \in K_+ \setminus U_+$. By the same argument, we have that there exists $d_- > 0$ satisfying (3.2) for all $v \in K_- \setminus U_-$.

Hence we consider the initial value problem of the form

$$(3.5) \quad \begin{aligned} \frac{du}{dt} + Lu &= g(u) + h, & 0 < t < T, \\ u(0) &= u_0, \end{aligned}$$

where $u_0 \in H$. Let $h \in W^{1,1}(0, T; H)$ and $u_0 \in H$. Then since g is Lipschitz continuous, the problem (3.5) has a unique solution $u \in W^{1,\infty}(0, T; H)$ such that $u(t) \in V$, a.e. $(0, \infty)$. (See, for example, [4, 5 or 6].) Noting that $du/dt \in L^\infty(0, T; H)$ and

$$\Phi(u(t)) - \Phi(u(s)) = \int_s^t \langle \Phi'(u(\tau)), u'(\tau) \rangle d\tau = \int_s^t \left\langle h(\tau) - \frac{du}{dt}, \frac{du}{dt} \right\rangle d\tau$$

for $0 < s \leq t < T$, we can see that the mapping $t \rightarrow \Phi(u(t))$ is continuous.

For each $h \in W^{1,\infty}(0, T; H)$ and $w \in H$, we set $T_h w = u(T)$, where u is the solution of (3.5) with $u_0 = w$. Then the mapping $T_h: H \rightarrow H$ is continuous. (See also [6].)

Lemma 6. *There exists $M > 0$ such that $T_h w \in K_\pm$, for all $w \in K_\pm$ and all $h \in W^{1,\infty}(0, T; H)$ with $\sup\{|h(t)|: t \in (0, t)\} \leq M$.*

Proof. By Lemma 5, it is easy to verify that there exist a positive number $M > 0$ and a negative number $c_* > c$ such that

$$(3.6) \quad \|Lv - g(v)\|_* \geq M \quad \text{for all } v \in W_\pm.$$

where $W_\pm = \{v \in V: \Phi(v) < c_*, P_2 v \in (\alpha_\pm^*, \beta_\pm^*)\} \setminus U_\pm$. Let $w \in K_+$ and $h \in W^{1,\infty}(0, T; H)$ such that $\sup\{|h(t)|: t \in (0, T)\} \leq M$. Let $u: (0, T) \rightarrow V$ be the solution of the problem (3.5) with $u_0 = w$. To prove Lemma 6, it is sufficient to show that $\Phi(u(t)) \leq c$ for all $t \in (0, T)$. Suppose that $\Phi(u(t)) \leq c$ on $(0, s)$ and $\Phi(u(s)) = c$ for some $0 \leq s < T$. Then we have that $u(s) \in K_+ \setminus U_+$. Since the mapping $t \rightarrow \Phi(u(t))$ is continuous, we find that there exists $s^* \in (s, T)$ such that $u(t) \in W_+$ on (s, s^*) . Then from (3.6), we find that

$$\begin{aligned} & \Phi(u(t)) - \Phi(u(s)) \\ &= \int_s^t \langle \Phi'(u(\tau)), u'(\tau) \rangle d\tau \\ &= \int_s^t \langle Lu(\tau) - g(u(\tau)), -(Lu(\tau) - g(u(\tau))) + h(\tau) \rangle d\tau \\ &\leq \int_s^t (|Lu(\tau) - g(u(\tau))|^2 + |Lu(\tau) - g(u(\tau))| |h(\tau)|) d\tau \\ &\leq \int_s^t (-\|Lu(\tau) - g(u(\tau))\|_*^2 + \|Lu(\tau) - g(u(\tau))\|_* |h(\tau)|) d\tau \\ &\leq -M^2(t-s) + M \cdot \sup\{|h(t)|: t \in (0, T)\}(t-s) \leq 0 \end{aligned}$$

for all $t \in (s, s^*)$. Therefore $\Phi(u(t)) \leq c$ for all $t \in (s, s^*)$. Thus we have shown that $\Phi(u(t)) \leq c$ on $(0, T)$. This completes the proof.

We can now complete

Proof of Theorem. Let $h \in W^{1,\infty}(0, T; H)$ such that $\sup\{|h(t)|: t \in (0, T)\} \leq M$. From the definition of T_h , the fixed points of T_h are the periodic

solutions of (P). Here we consider the mapping $ET_h Q_+ : \tilde{K}_+ \rightarrow \tilde{K}_+$. From the observation above, $T_h Q_+$ is continuous with respect to the norm of H . Then since $\tilde{K}_+$ is compact and convex in H , $T_h Q_+$ has a fixed point $w_+ \in \tilde{K}_+$. Noting that $Q_+ w_+ \in K_+ \subset \{v \in V : \Phi(v) \leq c\}$, we have by Lemma 6 that $w_+ = T_h Q_+ w_+ \in K_+$. Then $w_+ = Q_+ w_+$ and $w_+ = T_h w_+$. Therefore the solution u_+ of the problem (3.5) with the initial value $u_0 = w_+$ is a T -periodic solution. By the same argument, we have that there exists a solution u_- of (P) satisfying $u_-(0) \in K_-$. Then since $K_+ \cap K_- = \{\phi\}$, the problem (P) has at least two solutions.

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