A SEMIGROUP TREATMENT OF A ONE DIMENSIONAL NONLINEAR PARABOLIC EQUATION

HISAMITSU SERIZAWA

(Communicated by John B. Conway)

ABSTRACT. In this paper we study the existence, uniqueness and differentiability of solutions (in a certain generalized sense) for the nonlinear parabolic equation

$$u_t = u_{xx} - F(u, u_x) \quad (0 < x < 1, t > 0),$$

under the maximal monotone boundary conditions:

$$(-1)^i u_x(t, i) \in \beta_i(u(t, i)), \quad t > 0, \quad i = 0, 1.$$

1. INTRODUCTION AND SUMMARY OF RESULTS

We consider the following initial boundary value problem:

$$(P) \quad \left\{ \begin{array}{l} u_t = u_{xx} - F(u, u_x), \quad t > 0, \quad 0 < x < 1 \\ u_x(t, 0) \in \beta_0(u(t, 0)), \quad -u_x(t, 1) \in \beta_1(u(t, 1)), \quad t > 0 \\ u(0, x) = u_0(x), \quad 0 \leq x \leq 1, \end{array} \right\}$$

from the viewpoint of the theory of nonlinear semigroups.

Throughout this paper we assume that

- (β) β_i is a maximal monotone graph in $R \times R$ with $0 \in \beta_i(0)$, $i = 0, 1$,
- (F0) $F: R \times R \rightarrow R$ is a continuous function such that $F(0, 0) = 0$ and $F(\cdot, p): R \rightarrow R$ is nondecreasing for each $p \in R$.

Choose $C[0, 1]$ as the Banach space associated with (P) and define an operator A in $C[0, 1]$ by

$$(1.1) \quad \begin{aligned} D(A) &= \{u \in C^2[0, 1]; \quad (-1)^i u'(i) \in \beta_i(u(i)), i = 0, 1\} \\ Au &= u'' - F(u, u'), \quad u \in D(A). \end{aligned}$$

Then A is dissipative in $C[0, 1]$ and the problem (P) can be viewed as an evolution equation:

$$(1.2) \quad du/dt = Au(t), \quad t > 0, \quad u(0) = u_0 \text{ in } C[0, 1].$$

Received by the editors June 15, 1988.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 47H20, 47H06.

Key words and phrases. Nonlinear boundary value problem, m -dissipative operator.

If A is m -dissipative in $C[0, 1]$, then for every $u_0 \in \overline{D(A)}^{C[0,1]}$ the Crandall-Liggett theorem [2] provides us with the unique solution (in a certain generalized sense) of (1.2) represented by the exponential formula:

$$(1.3) \quad u(t, \cdot) = \lim_{\lambda \rightarrow 0+} (I - \lambda A)^{-[t/\lambda]} u_0 (\equiv \exp(tA)u_0), \quad t \geq 0.$$

We shall call $u(t, \cdot)$ the semigroup solution of (P).

The purpose of this paper is to give conditions on $F(u, p)$ under which the operator A defined by (1.1) is m -dissipative in $C[0, 1]$.

Theorem. *Let (β) and (FO) hold, and assume in addition that one of the following conditions holds:*

(F1) $|F(u, p)| \leq \zeta(u) \cdot \eta(|p|)$ for all $(u, p) \in R \times R$, where $\zeta(\cdot): R \rightarrow [0, \infty)$ is continuous, and $\eta(\cdot): [0, \infty) \rightarrow (0, \infty)$ is continuous, nondecreasing and $\int_0^\infty \frac{r}{\eta(r)} dr = \infty$.

(F2) $F_u: R \times R \rightarrow R$ is continuous and $F(0, p) = 0$ for all $p \in R$.
Then the operator A defined by (1.1) is m -dissipative in $C[0, 1]$.

Our result is closely related to the work of Konishi [5], in which he treated the existence, uniqueness and differentiability of semigroup solutions for the equation $u_t = u_{xx} - F(u_x)$ under the periodic boundary conditions. By almost the same methods in [5], we can examine the differentiability of semigroup solutions of (P).

Let us define the operator $\tilde{A}$ in $L^\infty(0, 1)$ by

$$(1.4) \quad \begin{aligned} D(\tilde{A}) &= \{u \in C^1[0, 1]; u'' \in L^\infty(0, 1), (-1)^i u'(i) \in \beta_i(u(i)), i = 0, 1\}, \\ \tilde{A}u &= u'' - F(u, u'), u \in D(\tilde{A}). \end{aligned}$$

Proposition. *Let the hypothesis of the theorem be satisfied. For each $u_0 \in D(\tilde{A})$ let $u(t) = \exp(tA)u_0$, $t \geq 0$ be the semigroup solution of (P). Then we have the following:*

- (i) $u(t) \in D(\tilde{A})$ for $t \geq 0$.
- (ii) $u \in C([0, \infty); C^1[0, 1])$.
- (iii) $u: [0, \infty) \rightarrow L^\infty(0, 1) = L^1(0, 1)^*$ is weakly* continuously differentiable and

$$w^* - (d/dt)u(t) = \tilde{A}u(t) \text{ in } L^\infty(0, 1) \text{ for } t \geq 0.$$

2. PRELIMINARIES

In what follows, by $C[0, 1]$ we mean the Banach space of all real valued continuous functions on $[0, 1]$ with the supremum norm $\|\cdot\|$. $C[0, 1]$ is a closed subspace of the Banach space $L^\infty(0, 1)$ with the norm $\|u\|_\infty = \text{ess sup}\{|u(x)|; x \in (0, 1)\}$, and $\|u\|_\infty = \|u\|$ for all $u \in C[0, 1]$. Similarly, let $C^1[0, 1]$ denote the Banach space of all continuously differentiable functions on $[0, 1]$ with the norm $\|u\|_1 = \max\{\|u\|, \|u'\|\}$.

An operator $A: D(A) \subset X \rightarrow X$ in a Banach space $(X, \|\cdot\|)$ is called dissipative if

$$\|u - v\| \leq \|u - v - \lambda(Au - Av)\| \quad \text{for all } u, v \in D(A)$$

and $\lambda > 0$.

A dissipative operator A in X is said to be m -dissipative if $R(I - \lambda A) = X$ for every, or equivalently, for some $\lambda > 0$; here $R(I - \lambda A)$ denotes the range of $I - \lambda A$.

A subset $\beta \subset R \times R$ is called monotone if $(u_1 - u_2)(v_1 - v_2) \geq 0$ whenever $v_i \in \beta(u_i)$, $i = 1, 2$. A monotone set not properly contained in any other monotone set is called maximal monotone. For more details on this material see [1].

3. PROOF OF THEOREM

We begin with the following lemmas.

Lemma 1. *Let (β) and (F0) hold. Then the operator A defined by (1.1) is dissipative in $C[0, 1]$.*

Proof. Let $u, v \in D(A)$ and let $\|u - v\| = |u(x_0) - v(x_0)|$, $x_0 \in [0, 1]$. Then we have

$$(3.1) \quad u'(x_0) - v'(x_0) = 0, \quad (u(x_0) - v(x_0))(u''(x_0) - v''(x_0)) \leq 0,$$

by maximum principle arguments and by the monotonicity of β_i , $i = 0, 1$. From this and (F0) the assertion follows easily.

The following lemma is a slight modification of [4, Chapter 12, Lemma 5.1].

Lemma 2. *Let $\eta: [0, \infty) \rightarrow (0, \infty)$ be a continuous and nondecreasing function satisfying $\int_0^\infty (r/\eta(r)) dr = \infty$, and let $R \geq 0$, $K \geq 0$, $C \geq 0$ be real numbers.*

Then there exists a number M (depending only on η , R , K , C) with the following property:

If $u \in C^2[0, 1]$, $|u(x)| \leq R$ and $|u''(x)| \leq K + C\eta(|u'(x)|)$, $x \in [0, 1]$, then $|u'(x)| \leq M$ for all $x \in [0, 1]$.

Proof of Theorem. By Lemma 1 it suffices to show $R(I - A) = C[0, 1]$. Define the operator A_0 in $C[0, 1]$ by $A_0 u = u''$ for $u \in D(A)$. Then A_0 is m -dissipative in $C[0, 1]$ and $(I - A_0)^{-1}: C[0, 1] \rightarrow C^1[0, 1]$ is continuous and compact. (See [3, Lemma 4.3] or [7, Proposition 1].)

Fix an arbitrary $f \in C[0, 1]$ and solve the equation

$$(3.2) \quad u = (I - A_0)^{-1}(f - F(u, u')).$$

Then the solution of (3.2) satisfies $u - Au = f$.

For $m = 1, 2, \dots$ let

$$B_m v = \begin{cases} f - F(v, v') & \text{if } \|v\|_1 \leq m, \\ f - F(mv/\|v\|_1, mv'/\|v\|_1) & \text{if } \|v\|_1 > m. \end{cases}$$

$B_m: C^1[0, 1] \rightarrow C[0, 1]$ is continuous and uniformly bounded. Consequently $T_m = (I - A_0)^{-1} B_m: C^1[0, 1] \rightarrow C^1[0, 1]$ is continuous, compact and uniformly bounded. Hence T_m maps some closed ball in $C^1[0, 1]$ into itself, and so by the Schauder's fixed point theorem we get a fixed point

$$(3.3) \quad u_m \in D(A); \quad u_m = (I - A_0)^{-1} B_m u_m, \quad m = 1, 2, \dots$$

If there is an m_0 such that $\|u_{m_0}\|_1 \leq m_0$ then u_{m_0} satisfies (3.2), and we complete the proof. Suppose, for contradiction, that $\|u_m\|_1 > m$ for all m . Then (3.3) can be rewritten as

$$(3.4) \quad u_m - u_m'' + F(mu_m/\|u_m\|_1, mu_m'/\|u_m\|_1) = f, \\ m = 1, 2, \dots$$

Let x_0 be any point in $[0, 1]$ satisfying $\|u_m\| = |u_m(x_0)|$. Then, using (3.1) with $u = u_m$ and $v = 0$ we get $u_m'(x_0) = 0$, $u_m(x_0)u_m''(x_0) \leq 0$. Plugging this into (3.4) gives

$$\|u_m\|^2 + F(mu_m(x_0)/\|u_m\|_1, 0)u_m(x_0) \leq \|f\| \|u_m\|.$$

From this and (F0) we obtain

$$(3.5) \quad \|u_m\| \leq \|f\| \quad \text{for all } m.$$

On the other hand, if F satisfies the condition (F1) then it follows from (3.4) and (3.5) that

$$|u_m''(x)| \leq 2\|f\| + C\eta(|u_m'(x)|) \quad \text{for all } x \in [0, 1],$$

where $C = \max\{\zeta(u); |u| \leq \|f\|\}$. Hence, it follows from Lemma 2 that there exists a constant M such that

$$(3.6) \quad \|u_m'\| \leq M \quad \text{for all } m.$$

Thus, by (3.5) and (3.6), we have

$$\|u_m\|_1 \leq \max\{\|f\|, M\} \quad \text{for all } m,$$

which is the desired contradiction.

Next suppose that F satisfies the condition (F2). Multiplying (3.4) by u_m'' and integrating over $[0, 1]$ we obtain

$$- [u_m u_m']_0^1 + \int_0^1 u_m'^2 dx + \int_0^1 u_m''^2 dx \\ - \int_0^1 F(mu_m/\|u_m\|_1, mu_m'/\|u_m\|_1) u_m'' dx = - \int_0^1 f u_m'' dx.$$

Let $G(u, p) = \int_0^p F(u, r) dr$. Then, by using (F0) and (F2), we can check easily that $G(u, p)up \geq 0$ and $G_u(u, p)p \geq 0$ for all $u, p \in R$. Hence by (β) we

have

$$\begin{aligned} & \int_0^1 F(mu_m/\|u_m\|_1, mu'_m/\|u_m\|_1) u''_m dx \\ &= m^{-1} \|u_m\|_1 [G(mu_m/\|u_m\|_1, mu'_m/\|u_m\|_1)]_0^1 \\ & \quad - \int_0^1 G_u(mu_m/\|u_m\|_1, mu'_m/\|u_m\|_1) u'_m dx \leq 0. \end{aligned}$$

From this and the monotonicity of β_i , $i = 0, 1$, we see that

$$\int_0^1 u''_m{}^2 dx \leq - \int_0^1 f u''_m dx \leq \frac{1}{2} \int_0^1 (f^2 + u''_m{}^2) dx,$$

and hence

$$\int_0^1 u''_m{}^2 dx \leq \int_0^1 f^2 dx \leq \|f\|^2.$$

Since $u'_m(x) = \int_{x_0}^x u''_m(r) dr$, we can conclude that

$$(3.7) \quad \|u'_m\| \leq \|f\|.$$

Thus, by (3.5) and (3.7), we have

$$\|u_m\|_1 \leq \|f\| \quad \text{for all } m,$$

which is also the desired contradiction. This completes the proof.

Remark. The Theorem is true if we assume (β) , (F0) and one of the following conditions:

- (F3) $F_u: R \times R \rightarrow R$ is continuous, $F(u, p) \geq 0$ for all $(u, p) \in R \times R$ and $R(\beta_i) \subset [0, \infty)$, $i = 0, 1$.
- (F4) $F_u: R \times R \rightarrow R$ is continuous, $F(u, p) \leq 0$ for all $(u, p) \in R \times R$ and $R(\beta_i) \subset (-\infty, 0]$, $i = 0, 1$.
- (F5) $F_u: R \times R \rightarrow R$ is continuous and $\beta_i = 0$, $i = 0, 1$.

4. PROOF OF PROPOSITION

In a similar way as in [5] we can prove the proposition. However, for completeness, we give here an outline of the proof.

The operator $\tilde{A}$ defined by (1.4) is dissipative in $L^\infty(0, 1)$ and satisfies the range condition:

$$\begin{aligned} R(I - \lambda \tilde{A}) &\supset R(I - \lambda A) = C[0, 1] \supset \overline{D(A)}^{C[0, 1]} = \overline{D(\tilde{A})}^{L^\infty(0, 1)} \\ &= \{u \in C[0, 1]; u(i) \in \overline{D(\beta_i)}, i = 0, 1\} (\equiv D), \quad \lambda > 0. \end{aligned}$$

Thus $\tilde{A}$ generates a nonlinear contraction semigroup $\exp(t\tilde{A})$, $t \geq 0$ on D in the sense of Crandall–Liggett [2], which together with (1.3) implies

$$\exp(t\tilde{A})u_0 = \exp(tA)u_0 \quad \text{for all } t \geq 0, u_0 \in D.$$

Let $u_0 \in D(\tilde{A})$. Then

$$\begin{aligned} \|(I - \lambda A)^{-[t/\lambda]} u_0\| &\leq \|u_0\|, \\ \|\tilde{A}(I - \lambda A)^{-[t/\lambda]} u_0\|_\infty &\leq \|\tilde{A}u_0\|_\infty \quad \text{for each } \lambda > 0 \text{ and } t \geq 0. \end{aligned}$$

Define the operator Λ by

$$\begin{aligned} D(\Lambda) &= \{u \in C[0, 1]; u' \in L^\infty(0, 1)\}, \\ \Lambda u &= u' \quad \text{for } u \in D(\Lambda). \end{aligned}$$

Then, by the same way as in the proofs of (3.6) and (3.7), we obtain the estimate:

$$\|\Lambda(I - \lambda A)^{-[t/\lambda]} u_0\| \leq M,$$

where M is a constant independent of $\lambda > 0$ and $t \geq 0$. Hence it follows that

$$\|\Lambda^2(I - \lambda A)^{-[t/\lambda]} u_0\|_\infty \leq \|\tilde{A}u_0\|_\infty + \max\{|F(u, p)|; |u| \leq \|u_0\|, |p| \leq M\}.$$

From these estimates we see that

$$\begin{aligned} s - \lim_{\lambda \rightarrow 0+} \Lambda(I - \lambda A)^{-[t/\lambda]} u_0 &= \Lambda \exp(tA)u_0 \quad \text{in } C[0, 1], \\ w^* - \lim_{\lambda \rightarrow 0+} \tilde{A}(I - \lambda A)^{-[t/\lambda]} u_0 &= \tilde{A} \exp(tA)u_0 \quad \text{in } L^\infty(0, 1) \end{aligned}$$

for each $t \geq 0$.

Furthermore, we can check that

$$\begin{aligned} \exp(\cdot A)u_0 &\in C([0, \infty); C^1[0, 1]) \text{ and the function} \\ t \in [0, \infty) &\rightarrow \tilde{A} \exp(tA)u_0 \in L^\infty(0, 1) \end{aligned}$$

is weakly* continuous. Thus, by using the estimate due to Ôharu [6, (8)], we can conclude that

$$\exp(tA)u_0 - u_0 = w^* - \int_0^t \tilde{A} \exp(sA)u_0 ds \quad \text{in } L^\infty(0, 1),$$

$t \geq 0$, from which the assertion follows.

REFERENCES

1. V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff International Publ., Leyden, The Netherlands, Bucuresti, 1976.
2. M. G. Crandall and T. M. Liggett, *Generation of semigroups of nonlinear transformations on general Banach spaces*, Amer. J. Math. **93** (1971), 265–298.
3. J. A. Goldstein and C. Y. Lin, *Singular nonlinear parabolic boundary value problems in one space dimension*, J. Differential Equations **68** (1987), 429–443.
4. P. Hartman, *Ordinary differential equations*, Wiley-Interscience, New York, 1964.
5. Y. Konishi, *On $u_t = u_{xx} - F(u_x)$ and the differentiability of the nonlinear semigroup associated with it*, Proc. Japan Acad. **48** (1972), 281–286.
6. S. Ôharu, *A note on the generations of nonlinear semigroups in a locally convex space*, Proc. Japan Acad. **43** (1967), 847–851.
7. H. Serizawa, *M-Browder-accretiveness of a quasilinear differential operator*, Houston J. Math. **10** (1984), 147–152.