ON THE MONOTONICITY OF THE PERMANENT

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ABSTRACT. Let Ω_n denote the set of all $n \times n$ doubly stochastic matrices and let $J_n = [1/n]_{n \times n}$. For $A \in \Omega_n$, if $f_A(t) = \text{per}((1-t)J_n + tA)$ is a nondecreasing function of t on $[0, 1]$, we say that the monotonicity of permanent (abb. MP) holds for A . Friedland and Minc [3] proved MP for $(nJ_n - I_n)/(n-1)$. In [6], Lih and Wang proposed a problem of determining whether MP holds for $J_{n_1} \oplus \cdots \oplus J_{n_k}$, $n_i > 0$.

In this note, we prove MP for $((mJ_m - I_m) \otimes sJ_s)/(m-1)s$, extending the result of Friedland and Minc, and give an affirmative answer to the Lih and Wang's question.

1. INTRODUCTION

Let Ω_n denote the set of all $n \times n$ doubly stochastic matrices and let K_n denote the $n \times n$ matrix of 1's. As usual, let I_n denote the identity matrix of order n and let $J_n = K_n/n$. For an $n \times n$ matrix A and for a real number t , let $A_t = (1-t)J_n + tA$ and let $f_A(t) = \text{per } A_t$, the permanent of A_t .

In [3], Friedland and Minc proved that $f_A(t)$ is a monotone increasing function of t on the closed unit interval $[0, 1]$ if either $A = I_n$ or $A = (K_n - I_n)/(n-1)$. The problem of finding matrices $A \in \Omega_n$ for which $f_A(t)$ is monotone increasing is found in [9, p. 158]. This property of $f_A(t)$ now is called the *monotonicity of permanent* (abb. MP) for A . In addition to those matrices which are permutation equivalent to I_n or $(K_n - I_n)/(n-1)$, several classes of matrices in Ω_n have turned out to satisfy MP (see [5, 8] for example). In [6], Lih and Wang proved MP for $J_s \oplus J_{n-s}$ and for

$$\begin{bmatrix} 0 & Y \\ Y^T & Z \end{bmatrix}$$

where Y is an $(n-t) \times t$ submatrix of J_t and $Z = (2-n/t)J_t$, and proposed a problem of determining whether MP holds for $J_{n_1} \oplus \cdots \oplus J_{n_k}$ where $n_i > 0$, $i = 1, \dots, k$.

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In this note we prove MP for $((K_m - P) \otimes K_s)/(m-1)s$ for any $m \times m$ permutation matrix P where $\otimes$ stands for the Kronecker product and $m \geq 2$, $s \geq 1$, extending Friedland and Minc' result, and give an affirmative answer to Lih and Wang's problem.

For a matrix A , let $A(i|j)$ denote the matrix obtained from A by deleting the row i and column j . Let D be a real valued function of Ω_n defined by

$$D(A) = \text{per } A - \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \text{per } A(i|j)$$

for $A \in \Omega_n$.

For $A = [a_{ij}] \in \Omega_n$, we compute

$$\frac{d}{dt} f_A(t) = \sum_{i=1}^n \sum_{j=1}^n \left(a_{ij} - \frac{1}{n} \right) \text{per } A_t(i|j)$$

and

$$D(A_t) = \frac{t}{n} \sum_{i=1}^n \sum_{j=1}^n \left(a_{ij} - \frac{1}{n} \right) \text{per } A_t(i|j)$$

so that

$$D(A_t) = \frac{t}{n} \frac{d}{dt} f_A(t).$$

Thus we have

Lemma 1. MP holds for $A \in \Omega_n$ if and only if $D(A_t) \geq 0$ for all t , $0 \leq t \leq 1$.

2. MP for $((K_m - I_m) \otimes K_s)/(m-1)s$.

To prove MP for $((K_m - I_m) \otimes P)/(m-1)s$ for an $m \times m$ permutation matrix P , we may assume that $P = I_m$. Let $n = ms$ so that

$$\frac{1}{(m-1)s} (K_m - I_m) \otimes K_s = \frac{1}{n-s} (K_n - K_s^{(m)})$$

where $K_s^{(m)}$ denotes the m -fold direct sum of K_s 's. Note that, in the case of $s = 1$,

$$\frac{1}{n-s} (K_m - I_m) \otimes K_s = \frac{1}{n-1} (K_n - I_n),$$

the matrix for which MP was proved by Friedland and Minc [3].

The following Lemma was conjectured by Sinkhorn [10] and proved by Bapat [1].

Lemma 2 [Sinkhorn-Bapat]. Let $A \in \Omega_n$ be such that $\text{per } A(i|j) = \text{per } A$ for all $i, j = 1, \dots, n$; then either $A = J_n$ or $A = \frac{1}{2}(I_n + P_n)$, up to permutations of rows and columns where P_n stands for the $n \times n$ full-cycle permutation matrix corresponding to the n -cycle $(12 \cdots n)$.

Let A be an $n \times n$ real matrix. We say that the positions (i, j) and (k, l) of A are equivalent if there exist permutation matrices P, Q such that $PAQ = A$

and the transformation $X \rightarrow PXQ$ takes the (i, j) -entry of X onto the (k, l) -entry of PXQ for all $n \times n$ matrix X .

It is clear that, if the positions (i, j) and (k, l) of A are equivalent then, $\text{per } A(i|j) = \text{per } A(k|l)$.

Lemma 3. *Let $m \geq 2$, $s \geq 1$ be integers and $n = ms$. For $(K_m - I_m) \otimes K_s := [x_{ij}]$, let $Z = \{(i, j) | x_{ij} = 0\}$. Let $A = [a_{ij}]$ be an $n \times n$ matrix such that all a_{ij} , $(i, j) \in Z$, are the same and all a_{kl} , $(k, l) \notin Z$, are the same. Then all the Z -positions of A are equivalent and all the off Z -positions of A are equivalent.*

Proof. For $p, q = 1, \dots, m$, let $T_{pq} = \{(i, j) | (p-1)s+1 \leq i \leq ps, (q-1)s+1 \leq j \leq qs\}$. Then $Z = \bigcup_{p=1}^m T_{pp}$. Clearly, all the positions of A in a single T_{pq} are equivalent.

To show the equivalence of the position $(1, s+1)$ and $(s+1, 1)$ of A , we can use the transformation $X \rightarrow PXP$ where

$$P = \begin{bmatrix} 0 & I_s \\ I_s & 0 \end{bmatrix} \oplus I_{n-2s}.$$

Thus we see that all the $T_{12} \cup T_{21}$ -positions of A are equivalent. In addition to that, this transformation tells us also that all the position of A in $T_{11} \cup T_{12}$ are equivalent.

In the case that $m \geq 3$, we see that the transformation $X \rightarrow QXQ$ with

$$Q = I_s \oplus \begin{bmatrix} 0 & I_s \\ I_s & 0 \end{bmatrix} \oplus I_{n-3s}$$

assures the equivalence of all the $T_{12} \cup T_{13}$ -positions of A . Repeating similar arguments a finite number of times, we see that all the Z -positions of A are equivalent and all the off Z -positions of A are equivalent. $\square$

Corollary. *Let $m \geq 2$, $s \geq 1$ be integers, $n = ms$, and let $A = (1/n - s)(K_m - I_m) \otimes K_s$. Then, for each $t \in [0, 1]$, there exist real numbers λ_t, μ_t such that*

$$(1) \quad \text{per } A_t(i|j) - \text{per } A_t = \begin{cases} \lambda_t & \text{if } (i, j) \in Z \\ \mu_t & \text{otherwise.} \end{cases}$$

Expanding $\text{per } A_t$ along the first row, we get the following relationship between λ_t and μ_t :

$$s(1-t)\lambda_t + (n-s)(1-t+nt)\mu_t = 0.$$

Thus, if $t \neq 1$, we have

$$(2) \quad s(1-t)\lambda_t + (n-s+st)\mu_t = 0.$$

Lemma 4. *Let $m \geq 2$, $s \geq 1$ be integers, $n = ms$, and let $A = (1/n - s)(K_m - I_m) \otimes K_s$. Then, for each $t \in [0, 1)$,*

$$D(A_t) = \frac{t}{1-t} \mu_t,$$

where μ_t is the number defined in (1).

Proof. Taking account of (1), we have

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n \text{per } A_t(i|j) &= \left(\sum_{(i,j) \in Z} + \sum_{(i,j) \notin Z} \right) \text{per } A_t(i|j) \\ &= n^2 \text{per } A_t + |Z| \lambda_t + (n^2 - |Z|) \mu_t \\ &= n^2 \text{per } A_t + ns \lambda_t + n(n-s) \mu_t. \end{aligned}$$

Therefore, by (2),

$$\begin{aligned} D(A_t) &= \text{per } A_t - \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \text{per } A_t(i|j) \\ &= -\frac{1}{n} (s \lambda_t + (n-s) \mu_t) \\ &= \frac{t}{1-t} \mu_t. \quad \square \end{aligned}$$

Now we are ready to prove the following theorem which is an extended version of Friedland and Minc's result [3], of which we give here an elementary combinatorial proof.

Theorem 1. *Let $m \geq 2$, $s \geq 1$ be integers, $n = ms$. Then MP holds for $(1/n - s)(K_m - P) \otimes K_s$ for all $m \times m$ permutation matrices P .*

Proof. Let $A = (1/n - s)(K_m - I_m) \otimes K_s$. To prove the theorem, it suffices to prove the MP for A only. We need to show that $D(A_t) \geq 0$ for all $t \in (0, 1)$, or equivalently, that $\mu_t \geq 0$ for all $t \in (0, 1)$, by Lemmas 1 and 4.

Suppose that $\mu_d < 0$ for some $d \in (0, 1)$.

Since J_n is the unique matrix in Ω_n with the minimum permanent [2], MP holds for matrices in a sufficiently small neighborhood of J_n in Ω_n . Hence we can take a $c \in (0, d)$ such that $D(A_c) > 0$ so that $\mu_c > 0$. Then, by Mean Value Theorem, there exists a $t \in (c, d)$ such that $\mu_t = 0$ and hence also that $\lambda_t = 0$ by (2). But then $\text{per } A_t(i|j) = \text{per } A_t$ for all $i, j = 1, \dots, n$, even if $A_t \neq J_n$ and $A_t \neq \frac{1}{2}(I_n + P_n)$, contradicting Lemma 2.

Thus we have shown that $\mu_t > 0$ for all $t \in (0, 1)$ and the proof is completed. $\square$

3. MP FOR P.S.D. SYMMETRIC MATRICES IN Ω_n

In this section we give a very simple proof for MP for positive semi-definite (abb. p.s.d.) symmetric doubly stochastic matrices, which assures the MP for $J_{n_1} \oplus \dots \oplus J_{n_k}$ where n_i , $i = 1, \dots, k$, are positive integers.

The following Lemma can be found in [7].

Lemma 5 [Marcus and Merris]. *If $A \in \Omega_n$ is p.s.d. symmetric, then $D(A) \geq 0$.*

As a corollary of Lemma 5, we have the following

Theorem 2. *MP holds for any p.s.d. symmetric doubly stochastic matrix.*

Proof. If $A \in \Omega_n$ is p.s.d. symmetric, then so is A_t for all $t \in [0, 1]$, so that $D(A_t) \geq 0$ for all $t \in [0, 1]$ by Lemma 5. Now MP for any p.s.d. symmetric doubly stochastic matrix follows from Lemma 1. $\square$

Since $J_{n_1} \oplus \cdots \oplus J_{n_k}$, $n_i > 0$, is a p.s.d. symmetric matrix, we have the following result which is the answer to the Lih and Wang's problem in [6].

Corollary 1. *MP holds for $J_{n_1} \oplus \cdots \oplus J_{n_k}$, $n_i > 0$.*

Corollary 2 ([4]). *Let $A_1, \dots, A_k \in \Omega_2$. Then MP holds for $A_1 \oplus \cdots \oplus A_k$.*

Proof. Every matrix in Ω_2 is permutation equivalent to a p.s.d. symmetric matrix. Thus, for $A_1, \dots, A_k \in \Omega_2$, $A_1 \oplus \cdots \oplus A_k$ is permutation equivalent to a p.s.d. symmetric matrix. $\square$

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