SCATTERED SPACES AND THE CLASS MOBI

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ABSTRACT. We show that every regular scattered space with a point-countable base is an open and compact image of a scattered metacompact Moore space and, consequently, is an element of the minimal class of regular spaces containing all the scattered metric spaces and invariant under open and compact mappings. This gives a characterization of a subclass of the class MOBI.

Let MOBI denote the minimal class of regular spaces containing all the metric spaces and invariant under open and compact mappings (see [A]).

It is easy to observe that a regular space is in MOBI if and only if it can be obtained as an image of a metric space under a mapping which is a composition of a finite number of open and compact mappings with regular domains (see [B]).

The first examples of nonmetacompact spaces in MOBI were constructed in [Ch]. These examples have recently been improved by a construction in [BCh] of nonweakly θ -refinable space in MOBI.

A common feature of these examples is that the pathological spaces in MOBI are scattered.

Let MOBI (scattered) denote the class of all the regular spaces which can be obtained as images of scattered metric spaces under mappings which are compositions of a finite number of open and compact mappings with regular domains.

The aim of this note is to prove

Theorem. *For a regular space Y the following conditions are equivalent.*

- (a) Y is a MOBI (scattered),
- (b) Y is a scattered space in MOBI,
- (c) Y is a scattered space with a point-countable base.

1. INTRODUCTION TO THE PROOF

Recall that a space Z is said to be a scattered space if each closed subset F of Z contains points which are isolated in the subspace topology of F .

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Since open mappings, clearly, preserve the above property, the implication (a) $\Rightarrow$ (b) in the theorem is obvious.

A collection $\mathcal{B}$ of subsets of a space Z is said to be point-countable (point-finite) if, for each $z \in Z$, the set $\{B \in \mathcal{B} : z \in B\}$ is countable (finite).

If $\{B(i) : i \in I\}$ is an indexed collection, then we say that it is point-countable (point-finite) as an indexed collection if for each $z \in Z$, the set $\{i \in I : z \in B(i)\}$ is countable (finite).

It is well known that all spaces in MOBI have a point-countable base. Thus, in order to prove the theorem, we only have to show that the implication (c) $\Rightarrow$ (a) holds. In the proof, we shall use the notion of a neighbornet introduced in [J1]

Recall that a neighbornet in a space Z is a relation $V \subset Z \times Z$ such that, for each $z \in Z$, the set $V(z) = \{z' \in Z : \langle z, z' \rangle \in V\}$ is an open subset of Z containing z .

Clearly, a neighbornet V in Z can be identified with the indexed collection $\{V(z) : z \in Z\}$.

A neighbornet V in Z is called co-countable (co-finite) if, for each $z \in Z$, $V^{-1}(z) = \{z' \in Z : \langle z', z \rangle \in V\}$ is countable (finite) [J2].

If a collection $\{V(z) : z \in Z\}$ defines a neighbornet V in Z , then $V^{-1}(z) = \{z' \in Z : z \in V(z')\}$ and, consequently, the neighbornet V is co-countable (co-finite) if and only if the collection $\{V(z) : z \in Z\}$ is point-countable (point-finite) as an indexed collection.

In [J2], it was shown that each first-countable T_1 -space X with a co-finite neighbornet is an open finite-to-one image of a σ -discrete metric space.

Note that the property of being a scattered space is an inverse invariant of mappings with scattered fibers and that spaces which are compact and countable are scattered.

From the two preceding paragraphs, it follows that in order to prove our theorem, it is sufficient to show that for each regular scattered space Y with a point-countable base, there exists a regular first-countable space X with a co-finite neighbornet and an open and compact countable-to-one mapping f of X onto Y .

We shall construct X and f by induction. Our proof will generalize some ideas used to construct X and f for the space Y having certain additional properties (see [Ch, 1.2 and BCh, 1.8]).

Before starting the induction, recall that a sequence $\langle W_j : j \geq 0 \rangle$ of open subsets of space Z is said to be strictly decreasing if $\overline{W_{j+1}} \subset W_j$ for $j \geq 0$.

Clearly, a first-countable T_1 -space Z is regular if and only if each point of Z has a strictly decreasing base.

2. THE INDUCTIVE CONSTRUCTION

Let Y be a regular scattered space with a point-countable base $\mathcal{V}$.

By induction, one can construct a partition $\mathcal{P} = \{P_\alpha : \alpha \in \sigma\}$ of Y such that, for each $\beta \in \sigma$, $Y_\beta = \bigcup \{P_\alpha : \alpha \in \beta\}$ is open in Y and $P_\beta = \{y\}$, where y is an isolated point of $Y \setminus Y_\beta$.

For each $y \in Y$, fix a $\beta_y \in \sigma$ such that

$$(1) \quad y \in P_{\beta_y}$$

and a $V(y) \in \mathcal{V}$ such that

$$(2) \quad y \in V(y) \subset \overline{V(y)} \subset \{y\} \cup Y_{\beta_y}.$$

Since $\mathcal{V}$ is point-countable and the indexing $y \rightarrow V(y)$ is, by (2), one-to-one, it follows that the collection $\{V(y) : y \in Y\}$ defines a co-countable neighbor-net V in Y .

For each $y \in Y$, let $\langle U_j(y) : j \geq 0 \rangle$ be a base of y in Y such that

$$(3) \quad \langle U_j(y) : j \geq 0 \rangle \text{ is strictly decreasing}$$

and

$$(4) \quad U_0(y) = V(y).$$

We shall construct, by induction on $\beta \in \sigma + 1$, spaces X_β and mappings f_β of X_β onto Y_β . The mapping $f_\sigma = f$ of $X_\sigma = X$ onto $Y_\sigma = Y$ will be the mapping needed to show that Y is in MOBI (scattered).

In order to assure that X is a regular first-countable space with a co-finite neighbor-net, we shall, for $\beta \in \sigma + 1$, construct inductively, for each $x \in X_\beta$ a base $\mathcal{B}_\beta(x) = \langle B_\beta(x, j) : j \geq 0 \rangle$ of x in X_β and, to help constructing $\mathcal{B}_{\beta+1}(x)$, for each $y \in Y_\beta$ a sequence $\mathcal{G}_\beta(y) = \langle G_\beta(y, j) : j \geq 0 \rangle$ of open subsets of X_β containing $f_\beta^{-1}(y)$.

We shall require the following conditions to be satisfied for $\beta \in \sigma + 1$.

- (i $_\beta$) $f_\beta(B_\beta(x, j)) \subset U_j(f_\beta(x))$ for $x \in X_\beta$ and $j \geq 0$,
- (i $^*_\beta$) $f_\beta(G_\beta(y, j)) \subset U_j(y)$ for $y \in Y_\beta$ and $j \geq 0$,
- (ii $_\beta$) f_β is an open mapping of X_β onto Y_β and the fibers of f_β are compact, Hausdorff and countable,
- (iii $_\beta$) $\mathcal{B}_\beta(x)$ is a strictly decreasing base for x in X_β ,
- (iii $^*_\beta$) $\mathcal{G}_\beta(y)$ is a strictly decreasing in X_β sequence of open subsets of X_β containing $f_\beta^{-1}(y)$ for $y \in Y_\beta$,
- (iv $_\beta$) $\{B_\beta(x, 0) : x \in X_\beta\}$ is point-finite in X_β as an indexed collection,
- (iv $^*_\beta$) $\{G_\beta(y, 0) : y \in Y_\beta\}$ is point-finite in X_β as an indexed collection,
- (v $_\beta$) if $z \in P_\beta$, then $z \in X_{\beta+1}$, and the restriction of $f_{\beta+1}$ to $B_{\beta+1}(z, 0)$ is an open and compact mapping of $B_{\beta+1}(z, 0)$ onto $V(z)$.

Note, that (ii_β) assures that $X_β$ is a T_1 -space and so, by (iii_β), $X_β$ is regular and first-countable, while (iv_β) implies that $X_β$ has a co-finite neighborhet. The remaining conditions are needed to continue the inductive construction.

We start our induction by putting, for $β = 0$, $X_0 = Y_0 = \emptyset$.

Suppose that $γ = β + 1$, $f_β : X_β \rightarrow Y_β$ and the collections $\{\mathcal{B}_β(x) : x \in X_β\}$, $\{\mathcal{G}_β(y) : y \in Y_β\}$ are given and satisfy (i_β)-(iv_β^{*}). We shall construct the corresponding objects for $γ$, so that (i_γ)-(iv_γ^{*}) and (v_β) will be satisfied. The last condition will be used in the limit step.

Note that $Y_γ = Y_β \cup P_β$. Let $z \in Y$ be such that $P_β = \{z\}$.

In order to extend $f_β : X_β \rightarrow Y_β$ to a mapping onto $Y_γ = Y_β \cup \{z\}$, we put

$$X_γ = X_β \cup \{z\}$$

and take $f_γ$ to be the natural extension of $f_β$.

For $j \geq 0$, conditions (2)-(4) imply that $U_j(z) \subset Y_γ$ and we define

$$(5) \quad B_γ(z, j) = G_γ(z, j) = f_γ^{-1}(U_j(z)).$$

Moreover, we put $B_γ(x, j) = B_β(x, j)$ for $x \in X_β$ and $j \geq 0$ and $G_γ(y, j) = G_β(y, j)$ for $y \in Y_β$ and $j \geq 0$.

Clearly, (i_γ) and (i_γ^{*}) are satisfied. In particular, $f_γ$ is continuous. Condition (ii_γ) is a consequence of (ii_β) and (5), while (v_β) follows from (ii_γ) and (4).

Using the continuity of $f_γ$ and (3) it is easy to see that sequences $\mathcal{B}_γ(z)$ and $\mathcal{G}_γ(z)$ defined by (5) are strictly decreasing. Thus, in order to prove (iii_γ) ((iii_γ^{*})), it is sufficient to observe that the continuity of $f_γ$ and (i_γ) ((i_γ^{*})) imply, for $x \in X_β$ ($y \in Y_β$), that the closure of $B_γ(x, 0)$ ($G_γ(y, 0)$) in $X_γ$ does not contain z .

Since (iv_γ) and (iv_γ^{*}) follow from the corresponding inductive assumptions, the successor step of our construction is complete.

Suppose that $λ \in \sigma + 1$ is a limit ordinal.

For $z \in Y_λ$, let $γ_z = β_z + 1$ (see (1)). To simplify the notation, define $W(z)$ to be $B_{γ_z}(z, 0) \subset X_{γ_z}$ and let $e_z : W(z) \rightarrow V(z)$ be the restriction of $f_{γ_z}$ to $W(z)$ (see (v_{β_z})).

Put

$$X_λ' = X_λ' \cup Y_λ^*,$$

where

$$X_λ' = \bigoplus \{W(z) : z \in Y_λ\}$$

is assumed to be disjoint from $Y_λ^*$ which is a subset of $Y_λ$.

Let e be the combination of mappings e_z for $z \in Y_λ$. Clearly e is open and maps $X_λ'$ onto $Y_λ$. Moreover, for each $y \in Y_λ$,

$$e^{-1}(y) = \bigoplus \{e_z^{-1}(y) : z \in Y_λ \text{ and } y \in V(z)\}.$$

Since the neighborset V is co-countable in Y , it follows that the fibers of e are countable and that $e^{-1}(y)$ is not compact if and only if $V^{-1}(y) \cap Y_\lambda$ is infinite. In order to simplify notation, denote $V^{-1}(y) \cap Y_\lambda$ as $I(y)$.

Let $Y_\lambda^* = \{y \in Y_\lambda : I(y) \text{ is infinite}\}$ and define f_λ to be the natural extension of e to X_λ . The points of Y_λ^* will be used to compactify the noncompact fibers of e .

For $x \in W(z) \subset X'_\lambda$ and $j \geq 0$, define

$$(6) \quad B_\lambda(x, j) = W(z) \cap B_{\gamma_z}(x, j).$$

Clearly, the sets generate the topology of X'_λ .

For $y \in Y_\lambda \setminus Y_\lambda^*$ and $j \geq 0$, define

$$(7) \quad G_\lambda(y, j) = \bigcup \{W(z) \cap G_{\gamma_z}(y, j) : z \in I(y)\}.$$

Finally, for each $y \in Y_\lambda^*$, fix a one-to-one enumeration $k_y : I(y) \mapsto \omega$ of the infinite set $I(y)$ and define, for $j \geq 0$,

$$(8) \quad B_\lambda(y, j) = \{y\} \cup \bigcup \{W(z) \cap G_{\gamma_z}(y, j + k_y(z)) : z \in I(y), k_y(z) \geq j\}$$

and

$$(9) \quad G_\lambda(y, j) = \{y\} \cup \bigcup \{W(z) \cap G_{\gamma_z}(y, j + k_y(z)) : z \in I(y)\}.$$

Consider X_λ with the topology generated by the sets described in (6) and (8). Note that our construction assures that the sets described in (7) and (9) are open subsets of X_λ containing the corresponding fibers of f_λ .

Clearly, (i_λ) and (i_λ^*) are satisfied. In particular, f_λ is continuous. Condition (ii_λ) is a consequence of our construction and the inductive assumptions (ii_{γ_z}) and (v_{γ_z}) .

Observe that, by (8), the sets $W(z)$ are clopen in X_λ . Thus, in order to prove that (iii_λ) and (iii_λ^*) hold, it is sufficient to use the inductive assumptions (iii_{γ_z}) and $(iii_{\gamma_z}^*)$ and the fact that for $y \in Y_\lambda^*$,

$$(*) \quad \overline{B_\lambda(y, 0)} \cap Y_\lambda^* = \{y\},$$

where the closure is taken in X_λ (note that $B_\lambda(y, 0) = G_\lambda(y, 0)$ for $y \in Y_\lambda^*$).

To prove $(*)$, take a $y' \in Y_\lambda^* \setminus \{y\}$ and find a $j \geq 0$ for which $U_j(y) \cap U_j(y') = \emptyset$. Let $j' \geq j$ be such that, for $z \in I(y) \cap I(y')$, $k_y(z) < j$ implies $k_{y'}(z) < j'$.

Suppose that $B_\lambda(y, 0)$ and $B_\lambda(y', j')$ have a common point $x \in W(z) \subset X'_\lambda$ and put $k = k_y(z)$, $k' = k_{y'}(z)$. Since $B_\lambda(y', j')$ intersects $W(z)$, (8) gives $k' \geq j'$ and the choice of j' assures that $k \geq j$. From (i_{γ_z}) , (8), (3) and the definition of e_z , we obtain $e_z(x) \in U_{0+k}(y) \cap U_{j'+k'}(y') \subset U_j(y) \cap U_j(y')$, which contradicts our choice of j . The contradiction completes the proof of $(*)$.

Finally, note that in order to prove (iv_λ) and (iv_λ^*) , it is sufficient to show that the collection $\{G_\lambda(y, 0) : y \in Y_\lambda\}$ is point-finite on X'_λ .

Suppose that $x \in W(z) \subset X'_\lambda$ and $x \in G_\lambda(y, 0)$ for a $y \in Y$. From our construction (see (7) and (9)), it follows that $z \in I(y) \subset V^{-1}(y)$, hence, (2) implies that $y \in V(z) \subset Y_{\gamma_z}$ and, by $(iv_{\gamma_z}^*)$, there is only finitely many such $y \in Y_{\gamma_z}$.

3. REMARKS

For a scattered space Z , let the standard partition of Z be the partition $\mathcal{L} = \{L_\alpha : \alpha \in \sigma\}$ such that, for each $\beta \in \sigma$, L_β is the nonempty set of all the isolated points of $Z \setminus \bigcup\{L_\alpha : \alpha \in \beta\}$ and define the height of Z to be the ordinal σ .

If Y is the space of natural numbers with the discrete topology and $Y_{n+1} = V(n) = \{0, \dots, n\}$, then the construction described in the previous section gives the space X of height 2 while the height of Y is 1.

A more efficient construction can be obtained by replacing the partition $\mathcal{P}$ of Y with the standard partition $\mathcal{L}$.

We shall describe modifications of the induction needed to construct f and X from $\mathcal{L}$ (see [Ch, 1.2]).

As before, we start by putting $X_0 = Y_0 = \emptyset$. We extend f_β to a mapping onto $Y_{\beta+1} = Y_\beta \cup L_\beta$ in two steps. First, for each $z \in L_\beta$, we construct a space $W(z)$ and an open and compact mapping e_z of $W(z)$ onto $V(z)$ as in the successor step described in §2. Next, we use the sets $W(z)$, for $z \in Y_{\beta+1}$, to construct $X_{\beta+1}$ as in the limit step described in §2.

The limit step remains unchanged.

It is easy to see that if the height of Y is 2, then the above construction is essentially the same as the proof of 1.2 in [Ch]. Thus, according to an observation from [Ch], if Y is not metacompact then the construction gives X which is not normal. The following result is a slight generalization of the observation from [Ch].

Proposition. *If $f : X \rightarrow Y$ is an open and compact mapping of a normal metacompact scattered space X onto Y and the height of X is 2, then Y is a metacompact space.*

Proof. Let $X = L_0 \cup L_1$, be the standard partition of X . Since f is an open mapping, it follows that $K_0 = f(L_0)$ consists of isolated points of Y (K_0 is the set of isolated points of Y but we do not use this fact).

Put $A_0 = L_1 \cap f^{-1}(K_0)$ and $A_1 = L_1 \setminus A_0$. By normality of X , there exist disjoint open sets U_0 and U_1 separating the closed sets A_0 and A_1 .

Clearly, U_1 is clopen in X . Moreover, $Y \subset f(U_1) \cup K_0$, hence $f(U_1)$ is clopen in Y and, in order to prove that Y is metacompact, it suffices to show that the restriction g of f to U_1 is finite-to-one.

Suppose that a fiber $C = g^{-1}(z)$ of g is infinite. Since C is a compact subset of X , it has to intersect both L_0 and L_1 . Thus, $z \in g(L_0) \subset f(L_0) = K_0$ and $L_1 \cap C \subset L_1 \cap g^{-1}(K_0) \subset L_1 \cap f^{-1}(K_0) = A_0 \subset U_0$, which contradicts the definition of g .

We do not know whether the above result holds if the height of X is greater than 2.

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