ON THE PRIMITIVITY OF GROUP RINGS OF AMALGAMATED FREE PRODUCTS

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ABSTRACT. In this paper we prove the primitivity of the group ring $F[G]$ where F is a field and G belongs to a certain class of free products of groups with amalgamation studied recently by the author. Thus our result generalizes a result of Formanek on free products of groups.

1. INTRODUCTION

Let V denote the class of groups which satisfy the following conditions (a), (b) and (c):

(a) $G = (A * B; K = \varphi(H))$ is the free product of two groups A and B with the subgroup H and K amalgamated.

(b) H does not contain the verbal subgroup $A(X^2)$ of A and K does not contain the verbal subgroup $B(X^2)$ of B .

Consequently, G contains a copy of F_2 freely generated by $x = aba$ and $y = bab$, where $a^2 \notin H$ and $b^2 \notin K$ ([1], Lemma 3.2).

(c) $(a^{-1}Ha) \cap H = \{e\} = (b^{-1}Kb) \cap K$.

For example, if

$$A = \langle a, c; a^3, c^2, (ac)^2 \rangle \quad \text{and} \quad B = \langle b, d; b^3, d^2, (bd)^2 \rangle,$$

the symmetric group on three objects, then the free product of A and B with the cyclic groups generated by c and d amalgamated is a group in V .

Let F be a field, G a group and $F[G]$ the group ring of G over F . If $G = \mathbf{Z} * \mathbf{Z}$ or $G_1 * G_2$, where $\mathbf{Z}$ is the infinite cyclic group and G_1, G_2 are not both of order 2, then it is known that $F[G]$ is primitive [4, 2]. In this paper, we show that $F[G]$ is primitive if G is in V , thus generalizing the results of [4 and 2], except when $G = G_1 * G_2$ where G_1 or G_2 has exponent 2.

2. NOTATION AND DEFINITIONS

Let $G = (A * B; K = \varphi(H))$. Suppose specific right coset representative systems for $A(\text{mod } H)$ and $B(\text{mod } K)$ have been selected. Then any element g in

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G can be represented uniquely as a product $hc_1c_2\cdots c_r$, called the *reduced form* of g , where h is in H , c_i is not in H , c_i is a representative from $A(\text{mod } H)$ or $B(\text{mod } K)$ and c_i, c_{i+1} are not both in A or both in B . The non-negative integer r is called the (representative) length $l(g)$ of g . If $g = hc_1c_2\cdots c_r$ is in the reduced form, then g is said to begin with c_1 and end with c_r . If c_1 is in A , we say that g has A -form. We define $-A$, $-B$, and $B-$ forms for elements of G similarly. If c_1 is in A and c_r is in B , we say that g has AB form. We define AA , BB , and BA forms similarly.

An ideal I in the ring R is called (right) *primitive* if it is the largest ideal contained in some maximal right ideal M , thus

$$I = R \cdot M = \{r \in R | Rr \subset M\}.$$

This definition is not symmetrical, and it is known that right primitivity does not imply left primitivity. Nonetheless, the attribute “right” is usually omitted. A ring is called (right) *primitive* if 0 is a primitive ideal.

A module A_R is called *faithful* if, for any $0 \neq r \in R$, $Ar \neq 0$. The following characterization of primitive rings is often taken as their definition.

Theorem (Jacobson). *The ring R is primitive if and only if there exists a faithful irreducible module A_R .*

Proof. See [3, p. 52].

3. PRIMITIVITY

Theorem 3.1. *Let G be a group in $\mathbf{V}$, and let R be an F -algebra. If R has no zero divisors and if $|R| \leq |G|$, an inequality of possibly infinite cardinals, then $R[G]$ is a primitive ring.*

Proof. For definiteness, we assume that $|A| \geq |B|$. By the defining condition (b) of $\mathbf{V}$, G contains a copy of F_2 and so it is infinite and its cardinality is not less than that of R , hence G and $R[G]$ must have the same cardinality. The proof is broken into two cases, depending on the finiteness or otherwise of $[A:H]$.

Case 1. The index of H in A is infinite.

Since A is infinite, $|A| \geq |B|$ and $|A*B| \geq |A*_HB|$, it is clear that $|A| = |G|$. Hence since $|G| \geq |R|$, we have $|A| = |R[G]|$. Let $\alpha: u \mapsto \alpha(u)$ be a fixed bijection from A onto $R[G]$, such that $\alpha(1) = 0$.

Let a, b, x, y be as in §2, and $w(u)$ an element of maximal length in the support of $\alpha(u)$, if $u \neq 1$. As in [2] we now construct a suitable linear form in $\alpha(u)$ that exhibits the element u in its structure.

For each $u \in H - \{1\}$, let $m(u)$ equal

$$\begin{array}{ll} au\alpha(u)a^{-1}b + bau\alpha(u)a^{-1} + 1 & \text{if } w(u) \in H \\ uab\alpha(u)b + uy\alpha(u) + 1 & \text{if } w(u) \text{ is of } AA \text{ form} \\ uy\alpha(u) + uab\alpha(u) + 1 & \text{if } w(u) \text{ is of } AB \text{ form} \\ ua\alpha(u)b + uba\alpha(u) + 1 & \text{if } w(u) \text{ is of } BA \text{ form} \\ uba\alpha(u) + ua\alpha(u)a + 1 & \text{if } w(u) \text{ is of } BB \text{ form.} \end{array}$$

For each $u \in A - H$, let $m(u)$ equal

$$\begin{array}{ll} u\alpha(u)b + bu\alpha(u) + 1 & \text{if } w(u) = 1 \\ uba\alpha(u)a^{-1}b + bub\alpha(u) + 1 & \text{if } w(u) \in H - \{1\} \\ ub\alpha(u)b + bub\alpha(u) + 1 & \text{if } w(u) \text{ is of } AA \text{ form} \\ bub\alpha(u) + ub\alpha(u) + 1 & \text{if } w(u) \text{ is of } AB \text{ form} \\ u\alpha(u)b + bu\alpha(u) + 1 & \text{if } w(u) \text{ is of } BA \text{ form} \\ bu\alpha(u) + u\alpha(u)u + 1 & \text{if } w(u) \text{ is of } BB \text{ form.} \end{array}$$

We note that

(i) Every maximal length element in the support of $m(u)$ is of form $A-$ or $BA-$.

(ii) There exist maximal length elements in the support of $m(u)$ which have $-A$ form and which have $-B$ form.

Let $s(u)$ be some nonzero element of $R[G]$, and w_0 an element of maximal length in its support. By (ii) there exists an element t_0 of maximal length in the support of $m(u)$ such that the product t_0w_0 is in reduced form. Since R is a ring with no zero divisors, t_0w_0 is contained in the support of $m(u)s(u)$. Hence we can deduce from (i) that

(iii) If $s(u) \neq 0$, then every maximal length element in the support of $m(u)s(u)$ has form $A-$ or $BA-$.

Hence the right ideal in $R[G]$ generated by $\{m(u); u \in A - \{1\}\}$ is proper. For otherwise we would get an equation of the form

$$1 = \sum_{j=1}^k m(u_j)s(u_j)$$

for some $s(u_j)$ ($j = 1, 2, \dots, k$), which contradicts (iii). Therefore, this right ideal is proper and so it is contained in a maximal ideal M . Consider the irreducible module $W = R[G]/M$. Let ρ be a nonzero element of $R[G]$ such that $I = R[G]\rho \subset M$. Since $I \neq 0$, it contains the element $\alpha(u)$ for some $u \in A - \{1\}$ and so by the definition of $\alpha(u)$, $\alpha(u) - 1 \in I \subset M$. Hence $1 \in M$, contrary to the choice of M as a maximal (proper) ideal. Therefore $R[G] \cdot M = 0$, and W is faithful. Thus $R[G]$ is a primitive ring.

Case 2. The index of H in A is finite.

Since $|A:H| \geq |B:K|$, it is clear that both $|G:H|$ and $|R[G]:R[H]|$ are countably infinite. Let $\{\alpha(n): n = 1, 2, 3, \dots\}$ be an enumeration of a right coset representative system for $R[G] \pmod{R[H]}$. Let a and b be as before so that $a^2 \notin H$ and $b^2 \notin K$. Let $w(n)$ be a group element of maximal length in the support of $\alpha(n)$ and we let $m(n)$ equal

$$\begin{aligned} (ab)^n a^2 \alpha(n) b + b(ab)^n a^2 \alpha(n) + 1 & \quad \text{if } w(n) = 1 \text{ or } w(n) \text{ is of } BA \text{ form} \\ (ab)^n a^2 b a \alpha(n) a^{-1} b + b(ab)^n a^2 b \alpha(n) + 1 & \quad \text{if } w(n) \in H - \{1\} \\ (ab)^n a^2 \alpha(n) a + b(ab)^n a^2 \alpha(n) + 1 & \quad \text{if } w(n) \text{ is of } BB \text{ form} \\ (ab)^n a^2 b \alpha(n) b + b(ab)^n a^2 b \alpha(n) + 1 & \quad \text{if } w(n) \text{ is of } AA \text{ form} \\ (ab)^n a^2 b \alpha(n) a + b(ab)^n a^2 b \alpha(n) + 1 & \quad \text{if } w(n) \text{ is of } AB \text{ form.} \end{aligned}$$

As in Case 1, we note firstly that every group element of maximal length in the support of $m(n)$ begins with $(ab)^n a^2$ or $b(ab)^n a^2$; secondly that there exist maximal length group elements in the support of $m(n)$ of the form $-A$ and of the form $-B$. From these we conclude as before that if $s(n) \neq 0$, then every maximal length group element in the support of $m(n)s(n)$ begins with either $(ab)^n a^2$ or $b(ab)^n a^2$. Thus $\sum_n m(n)R[G]$ is a proper ideal. This is extended to a maximal ideal M which contains no nonzero ideal of $R[G]$. As before we can conclude that $R[G]$ is primitive. This completes the proof.

Theorem 3.2. *Let $G \in \mathbf{V}$. If F is any field, then $F[G]$ is a primitive ring.*

Proof. It suffices to show that

$$\Delta(G) = \{g \in G: |G: C_G(g)| < \infty\} = \{1\}.$$

Let $g \in G - \{1\}$, by Lemma 3.3 of [1] there exists an integer k such that $(aba)^k g (aba)^k$ (when written in reduced form) begins and ends with a coset representative from $A \pmod{H}$. We can then alternately conjugate this new element by elements of $A - H$ and $B - K$ to obtain infinitely many conjugates of g . Thus $g \notin \Delta(G)$ and $\Delta(G) = 1$.

Let F be a given field and F_0 be its prime subfield. Then F_0 is countable, and so $|F_0| \leq |G|$. Taking $R = F_0$ in Theorem 3.1, we have that $F_0[G]$ is primitive. Finally since $\Delta(G) = 1$, the desired result follows by [5 or 6, p. 356].

Remark 3.3. Let S be a subgroup of a group T . A distinct pair $\{a_1, a_2\}$ of elements of T , neither of which are in S , is called a *blocking pair for S in T* if

$$(i) \quad a_i^\varepsilon a_j^\delta \notin S, \quad 1 \leq i, j \leq 2, \varepsilon = \pm 1, \delta = \pm 1, \text{ unless } a_i^\varepsilon a_j^\delta = e,$$

and

$$(ii) \quad \text{if } x \in S, \quad x \neq e, \text{ then}$$

$$a_i^\varepsilon x a_j^\delta \notin S, \quad 1 \leq i, j \leq 2, \varepsilon = \pm 1, \delta = \pm 1.$$

Let $G = A *_H B$ where $A \neq H \neq B$ and there exists a blocking pair $\{x_1, x_2\}$ for H in one of the factors of G , then it can be shown that $F[G]$ is primitive for any field F .

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