REFINABLE MAPS AND θ_n -CONTINUA

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ABSTRACT. Many properties of continua, e.g., irreducibility, span zero, and various kinds of aposyndesis and unicoherence, are preserved by refinable maps or their inverses. The purpose of this paper is to consider the extent to which the property of being a θ'_n -continuum is preserved by refinable maps or by inverses of refinable maps. One consequence of this study is to generalize results of the first author where the domain or range is a graph. Several of his results are corollaries of ours.

DEFINITIONS AND PRELIMINARIES

A *continuum* is a compact, connected, metric space and a θ_n -continuum [1, [4, p. 261] is one where the complement of any subcontinuum has at most n components. A θ'_n -continuum [5, p. 56] is a θ_n -continuum which admits a unique, monotone, upper semi-continuous decomposition, called the canonical decomposition, whose elements have void interior and whose quotient space is a graph. The elements of the decomposition are called *tranches*. The *order* of a tranche is its order considered as an element of the quotient space. If A is a subset of a continuum X , then $K(A) = \cap\{Q \mid Q \text{ is a subcontinuum of } X \text{ that contains } A \text{ in its interior}\}$ ([7, Theorem 2, p. 404]), and $T(A) = \{x \in X \mid \text{no subcontinuum in } X \setminus A \text{ contains } x \text{ in its interior}\}$ [4, pp. 261–262]. It is easily seen that $K(H) = T(H)$ for each subcontinuum H of a θ_n -continuum. It is also easily seen that $K(H) \setminus H$ has void interior for any subcontinuum H of a θ'_n -continuum. From this and [4, Theorem 1, p. 263], it follows that a continuum X is a θ'_n -continuum if and only if X is a θ_n -continuum for which $K(H) \setminus H$ has void interior for each subcontinuum H of X . Let $f: X \rightarrow Y$ mean f is a function from X onto Y . Suppose X and Y are continua, $\varepsilon > 0$, and $f: X \rightarrow Y$. Then f is an ε -map if f is a map, i.e., a continuous function, and $f^{-1}(y)$ has diameter less than ε for every $y \in Y$. A map $g: X \rightarrow Y$ is an

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ε -approximation of f , if $d(f(x), g(x)) < \varepsilon$ for every $x \in X$. If g is both an ε -map and an ε -approximation of f , then g is an ε -refinement of f . A map $f: X \rightarrow Y$ is refinable [2, p. 263], if for each $\varepsilon > 0$ there is an ε -refinement of f . If the ε -refinements of f can be taken to be homeomorphisms, then f is a near homeomorphism.

Let $f: X \rightarrow Y$ be a refinable map, and B be a closed subset of Y . For each positive integer n , let f_n be a $\frac{1}{n}$ -refinement of f such that $\{f_n^{-1}[B]\}$ converges. Then B' denotes $\lim f_n^{-1}[B]$. Observe that B' is not uniquely determined by f , since it depends on the choice of $\{f_n\}$. Heath (Ford) and Rogers [2, Theorem 1, p. 264] have proved that, for each subcontinuum B of Y , every B' is a subcontinuum of X such that $f(B') = B$ and $f^{-1}[\text{Int}(B)] \subseteq B'$, where $\text{Int}(B)$ is the interior of B . We use both of these results frequently. To avoid notational difficulties we occasionally use B^0 for $\text{Int}(B)$. If $f: X \rightarrow Y$ and decomposition spaces of X and Y are also being considered we use $f[A]$ for $\{f(x): x \in A\}$, if $A \subseteq X$, and we use $d[A, B]$ for $\inf\{d(x_1, x_2): x_1 \in A \text{ and } x_2 \in B\}$, if $A \subseteq X$ and $B \subseteq Y$.

1. FOCUS ON THE DOMAIN

In this section we consider refinable maps whose domains are θ_n -continua or θ'_n -continua. First we establish a lemma.

Lemma 1. Suppose $f: X \rightarrow Y$ is refinable and B is a closed subset of Y . Then $[K(B)]' \subseteq K(B')$, if these sets are determined by the same sequence of $\frac{1}{n}$ -refinements of f .

Proof. For $i = 1, 2, \dots$, let f_i be a $\frac{1}{i}$ -refinement of f such that $f_1^{-1}(B)$, $f_2^{-1}(B)$, $\dots$, and $f_1^{-1}(K(B))$, $f_2^{-1}(K(B))$, $\dots$, converge to B' and $[K(B)]'$, respectively. We wish to show that $x \notin [K(B)]'$ if $x \notin K(B')$. Suppose $x \notin K(B')$ and let H be a subcontinuum of X such that $B' \subseteq H^0 \subseteq H \subseteq X \setminus \{x\}$. Let ε be a positive number such that $d(x, H) > 2\varepsilon$ and $N_{2\varepsilon}(B') \subseteq H$. Let $m > \frac{1}{\varepsilon}$ be such that $f_m^{-1}(B) \subseteq N_\varepsilon(B')$. Then $B = f_m(f_m^{-1}(B)) \subseteq Y \setminus f_m(X \setminus H^0) \subseteq [f_m(H)]^0 \subseteq f_m(H)$, since $d(f_m^{-1}(B), X \setminus H^0) > \varepsilon$ and f_m is an ε -map. Hence $K(B) \subseteq f_m(H)$. Since f_m is an ε -map and $d(H, N_\varepsilon(x)) > \varepsilon$, we have $f_m^{-1}(K(B)) \subseteq f_m^{-1}(f_m(H)) \subseteq X \setminus N_\varepsilon(x)$, so $x \notin [K(B)]'$.

Corollary 2. If the continuum X has the property that $K(H) \setminus H$ has void interior for every subcontinuum H of X , and $f: X \rightarrow Y$ is a refinable map, then Y has this property for K also.

Proof. Let H be a subcontinuum of Y . Then $f^{-1}([K(H)]^0) \subseteq [K(H)]'$, $[K(H)]' \subseteq K(H')$ by Lemma 1, and therefore, since $K(H') \setminus H'$ has void interior, $f^{-1}((K(H))'^0) \subseteq H'$. So $[K(H)]^0 = f(f^{-1}((K(H))'^0)) \subseteq f(H') = H$. Hence $[K(H) \setminus H]^0 = \emptyset$.

The next two theorems show that the image of a θ'_n -continuum under a refinable map is a θ'_n -continuum and, although domain and image need not

be homeomorphic [3, Example 2, p. 148], the canonical decompositions into graphs are homeomorphic.

Theorem 3. *If X is a θ'_n -continuum and $f: X \rightarrow Y$ is a refinable map, then Y is a θ'_n -continuum, and, if L is a tranche in X , then $f[L]$ is contained in a tranche in Y . Hence, if X_D and Y_D are the canonical decompositions of X and Y , respectively, then f induces a map from X_D onto Y_D .*

Proof. From Corollary 2, it follows that Y has the property that $K(H) \setminus H$ has void interior for every subcontinuum H of Y , since X , as a θ'_n -continuum, has this property. So, to see that Y is a θ'_n -continuum, it suffices to show that Y is a θ_n -continuum. Suppose Y contains a subcontinuum H such that $Y \setminus H = \bigcup_{i=1}^{n+1} A_i$ where the A_i 's are mutually separated sets. Since $K(H) \setminus H$ has void interior, there exists a continuum R such that $H \subseteq \text{Int}(R) \subseteq R$ and $A_i \setminus R \neq \emptyset$, for $i = 1, \dots, n+1$. Let $A_i^* = A_i \setminus R$ for $i = 1, \dots, n+1$. For $k = 1, \dots$, let f_k be a $\frac{1}{k}$ -refinement of f such that $H' = \lim f_k^{-1}[H]$ and $R' = \lim f_k^{-1}[R]$ exist. Since X is a θ_n -continuum, $X \setminus R' = \bigcup_{m=1}^j B_m$, where each B_m is a component of $X \setminus R'$ and $j \leq n$. Let k_0 be chosen so that $\text{cl}(A_i^*) \setminus f_{k_0}[R'] \neq \emptyset$, for $i = 1, \dots, n+1$, and $d[f_{k_0}^{-1}(H), \bigcup_{m=1}^j B_m] > 1/k_0$. Then $\bigcup_{i=1}^{n+1} [\text{cl}(A_i^*) \setminus f_{k_0}[R']] \subseteq \bigcup_{m=1}^j f_{k_0}[B_m]$, and $f_{k_0}[B_m] \cap H = \emptyset$, for $m = 1, \dots, j$. Also, since each B_m is connected, $f_{k_0}[B_m]$ must be contained in one member of the collection of mutually separated sets $\{A_i | i = 1, \dots, n+1\}$. But $j < n+1$ so it follows that there exists an integer n_0 such that $A_{n_0}^* \setminus f_{k_0}[X] \neq \emptyset$, a contradiction, since f_{k_0} is surjective. It follows that Y is a θ'_n -continuum.

Let L be a tranche in X . Then by [4, Lemma 4, pp. 268, 269], and the equality of $K(A)$ and $T(A)$ in θ_n -continua, there exists a point $p \in L$ such that $K^n(p) = L$, where $K^1(p) = K(\{p\})$ and $K^m(p) = K(K^{m-1}(p))$ for $m = 2, 3, \dots$. Hosokawa has proved [6, Theorem 1'', p. 368] that $f[K(p)] \subseteq K(f(p))$, and his argument can be extended to subsets A of X so that $f[K(A)] \subseteq K[f(A)]$. Then $f[L] = f[K^n(p)] \subseteq K^n(f(p))$ by n applications of the extension of Hosokawa's result. Since Y is a θ'_n -continuum, $K(A)$ has void interior if A is a subcontinuum of Y with void interior. By n applications of this fact, $K^n(f(p))$ has void interior and hence must be contained in the tranche of Y containing $f(p)$. So $f[L]$ is also contained in this tranche. Thus the map $f_D = \eta_Y \circ f \circ \eta_X^{-1}$ is a map from X_D onto Y_D , where η_X and η_Y are the quotient maps from X onto X_D and Y onto Y_D , respectively.

Theorem 4. *The induced map f_D , of Theorem 3, from X_D onto Y_D is monotone and has the property that for each y_D in Y_D , $f_D^{-1}(y_D)$ contains no simple closed curve and contains at most one point not of order two. Hence, f_D is a near homeomorphism.*

Proof. Let $y_D \in Y_D$, and let $\mathcal{E}_1, \mathcal{E}_2, \dots$ be continua in Y_D such that $\{y_D\} = \bigcap_{i=1}^{\infty} \mathcal{E}_i^0 = \bigcap_{i=1}^{\infty} \mathcal{E}_i$, and $\mathcal{E}_1 \supseteq \mathcal{E}_1^0 \supseteq \mathcal{E}_2 \supseteq \mathcal{E}_2^0 \supseteq \mathcal{E}_3 \supseteq \dots$. Then $\eta_Y^{-1}(y_D) =$

$\bigcap_{i=1}^{\infty} \eta_Y^{-1}[\mathcal{E}_i^0] = \bigcap_{i=1}^{\infty} \eta_Y^{-1}[\mathcal{E}_i]$. It follows that $f^{-1}[\eta_Y^{-1}(y_D)] = f^{-1}[\bigcap_{i=1}^{\infty} \eta_Y^{-1}[\mathcal{E}_i^0]] = \bigcap_{i=1}^{\infty} f^{-1}[\eta_Y^{-1}[\mathcal{E}_i^0]] \subseteq \bigcap_{i=1}^{\infty} (\eta_Y^{-1}[\mathcal{E}_i])' \subseteq \bigcap_{i=1}^{\infty} f^{-1}[\eta_Y^{-1}[\mathcal{E}_i]] = f^{-1}[\eta_Y^{-1}(y_D)]$. So all of these sets are the same, in particular $f^{-1}[\eta_Y^{-1}(y_D)] = \bigcap_{i=1}^{\infty} (\eta_Y^{-1}[\mathcal{E}_i])'$. Since the latter set is the intersection of a nested sequence of continua (if the $\frac{1}{i}$ -refinements are chosen correctly), $f^{-1}[\eta_Y^{-1}(y_D)]$ is connected and, consequently, so is $\eta_X[f^{-1}[\eta_Y^{-1}(y_D)]] = (\eta_Y \circ f \circ \eta_X^{-1})^{-1}(y_D) = f_D^{-1}(y_D)$. This shows that $\eta_Y \circ f$ and f_D are monotone.

Now let us suppose that for some point T of Y_D , the continuum $f_D^{-1}(T)$ contains two points of order greater than 2 in the graph X_D . This implies that there are two tranches T_1 and T_2 of order greater than 2 as points of X_D and a subcontinuum I_0 of X irreducible from T_1 to T_2 such that $\eta_Y^{-1}(T) \supseteq f[T_1 \cup T_2 \cup I_0]$ and all of the tranches in $I_0 \setminus (T_1 \cup T_2)$ are of order 2 in X_D .

Let P_2 be the set of all points of X_D that are of order 2 in X_D . Let $\mathcal{U}$ be a subset of X_D (i.e., a collection of tranches in X) such that (1) $\mathcal{U} \supseteq X_D \setminus P_2$; (2) if $\mathcal{E}$ is a component of $(X_D \setminus \eta_X[I_0]) \cap P_2$ such that $\text{cl}(\mathcal{E})$ is an arc, then $\mathcal{U}$ contains one and only one point of $\mathcal{E}$; (3) if $\mathcal{E}$ is a component of P_2 such that $\text{cl}(\mathcal{E})$ is a simple closed curve, then $\mathcal{U}$ contains two and only two points of $\mathcal{E}$; and (4) T_1 and T_2 are the only members of $\mathcal{U}$ that intersect I_0 . Now, in addition to (4), each point of $X_D \setminus \mathcal{U}$ has order 2 in X_D , and the closures of any two components of $X_D \setminus \mathcal{U}$ are arcs that have no more than one common end point.

Let $L = (X \setminus I_0) \setminus (\bigcup \mathcal{U})$. Let $\{I_1, \dots, I_a\}$ be the collection of closures of all of those components of L that have a limit point in T_1 . Let $\{J_1, \dots, J_b\}$ be the collection of closures of all of those components of L that have a limit point in T_2 .

A construction similar to the construction of $\mathcal{U}$ can be used to select a finite collection $\mathcal{V}$ of points of Y_D such that $T \in \mathcal{V}$, each point of $Y_D \setminus \mathcal{V}$ has order 2 in Y_D , and the closure of each component of $Y_D \setminus \mathcal{V}$ is an arc.

In X , let $\delta = \frac{1}{2} \min\{d[U_1, U_2] : U_1 \in \mathcal{U}, U_2 \in \mathcal{U}, \text{ and } U_1 \neq U_2\}$. In Y , let γ be a positive number such that $N_{2\gamma}[T]$ is contained in the component of $[Y \setminus (\bigcup \mathcal{V})] \cup T$ that contains T . Let D_1 and D_2 be connected open subsets of X such that $T_1 \subseteq D_1 \subseteq N_\delta[T_1]$, $T_2 \subseteq D_2 \subseteq N_\delta[T_2]$, and $f[D_1 \cup D_2 \cup I_0] \subseteq N_\gamma[T]$. Note that $5\delta \leq d[T_1, T_2]$, so there is a point p in $I_0 \setminus \text{cl}(N_{2\delta}[T_1 \cup T_2])$ and $d[p, \text{cl}(D_1 \cup D_2)] > \delta$. Let $\varepsilon_1 = d[\text{cl } D_1, \text{cl}((X \setminus I_0) \setminus \bigcup_{i=1}^a I_i)]$ and $\varepsilon_2 = d[\text{cl } D_2, \text{cl}((X \setminus I_0) \setminus \bigcup_{i=1}^b J_i)]$. Let $\varepsilon_3 = \min\{d[I_i \setminus D_1, I_j] : 1 \leq i \neq j \leq a\}$ and $\varepsilon_4 = \min\{d[J_i \setminus D_2, J_j] : 1 \leq i \neq j \leq b\}$. Let $\varepsilon_5 = d[I_0 \setminus (D_1 \cup D_2), \text{cl}(X \setminus I_0)]$. Let $\varepsilon = \min\{\delta, \gamma, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\}$. Let f_ε be an ε -refinement of f . Then $f_\varepsilon[\text{cl}(D_1)] \cap f_\varepsilon[\text{cl}(D_2)] = \emptyset$, since $d[\text{cl}(D_1), \text{cl}(D_2)] \geq 3\delta \geq \varepsilon$.

It follows that either $f_\varepsilon[\text{cl}(D_1)]$ or $f_\varepsilon[\text{cl}(D_2)]$ does not intersect T in Y . To see this, assume both do intersect T . Since $d[I_0 \setminus (D_1 \cup D_2), \text{cl}(X \setminus I_0)] = \varepsilon_5 \geq \varepsilon$ and $d[p, \text{cl}(D_1 \cup D_2)] > \delta \geq \varepsilon$, it follows that $f_\varepsilon(p) \in f_\varepsilon[I_0] \setminus f_\varepsilon[\text{cl}(D_1 \cup D_2)] \subseteq f_\varepsilon[I_0 \setminus (D_1 \cup D_2)] \subseteq f_\varepsilon[I_0]$ and $f_\varepsilon[I_0 \setminus (D_1 \cup D_2)] \cap f_\varepsilon[\text{cl}(X \setminus I_0)] = \emptyset$. Hence,

$f_\varepsilon[I_0 \setminus (D_1 \cup D_2)]$ is a nonvoid subset of $\text{Int}(f_\varepsilon[I_0])$, and $f_\varepsilon[I_0] \setminus f_\varepsilon[\text{cl}(D_1 \cup D_2)]$ is a nonvoid open set. Since T has void interior, $f_\varepsilon[I_0] \setminus f_\varepsilon[\text{cl}(D_1 \cup D_2)]$ intersects some component C , of $Y \setminus (\bigcup \mathcal{V})$, whose closure intersects T . Since $\text{cl}(C)$ is irreducible from T to some member V_1 of $\mathcal{V}$, $f_\varepsilon[\text{cl}(D_1)]$ and $f_\varepsilon[\text{cl}(D_2)]$ are continua that intersect T , and $\{f_\varepsilon[I_0] \setminus f_\varepsilon[\text{cl}(D_1 \cup D_2)]\} \cap C$ is a nonvoid open set whose closure does not intersect V_1 , it follows that $f_\varepsilon[I_0 \cup \text{cl}(D_1 \cup D_2)] \cap C$ is an open and closed (relative to C) proper subset of C . That is not possible, since C is connected, so either $f_\varepsilon[\text{cl}(D_1)]$ or $f_\varepsilon[\text{cl}(D_2)]$ does not intersect T .

Since at least one of $f_\varepsilon[\text{cl}(D_1)]$ and $f_\varepsilon[\text{cl}(D_2)]$ does not intersect T , at least one of them neither intersects T nor cuts the other from T in the component of $(Y \setminus (\bigcup \mathcal{V})) \cup T$ that contains T . Assume without loss of generality that $f_\varepsilon[\text{cl}(D_1)]$ does not intersect T or cut $f_\varepsilon[\text{cl}(D_2)]$ from T in that set. Then there is a tranche V in $\mathcal{V}$ and a component $\mathcal{C}$ of $Y_D \setminus \mathcal{V}$ containing $\eta_Y[f_\varepsilon[\text{cl}(D_1)]]$ such that $\text{cl}(\mathcal{C})$ is an arc in Y_D with end points T and V . Now $f_\varepsilon[I_0] \cap V = \emptyset$, since $d[T, V] \geq 2\gamma \geq \varepsilon$. As we saw above, no point of $f_\varepsilon[I_0 \setminus (D_1 \cup D_2)]$ is a limit point of $Y \setminus f_\varepsilon[I_0]$. Hence the closed set $f_\varepsilon[I_0 \setminus (D_1 \cup D_2)]$ does not intersect the component of $\eta_Y^{-1}[\text{cl}(\mathcal{C}) \setminus f_\varepsilon[\text{cl}(D_1)]]$ that contains V . Now $f_\varepsilon[\text{cl}(D_1) \cup (I_0 \setminus D_2)]$ is a continuum containing, but not contained in, $f_\varepsilon[\text{cl}(D_1)]$, so any subcontinuum of $\eta_Y^{-1}[\text{cl}(\mathcal{C})]$ that intersects $f_\varepsilon[\text{cl}(D_1)]$ and the component of $\eta_Y^{-1}[\text{cl}(\mathcal{C}) \setminus f_\varepsilon[\text{cl}(D_1)]]$ that contains T also intersects $f_\varepsilon[\text{cl}(D_1) \cup (I_0 \setminus D_2)] \setminus f_\varepsilon[\text{cl}(D_1)] \subseteq f_\varepsilon[I_0 \setminus (D_1 \cup D_2)]$. But $f_\varepsilon[I_i] \setminus f_\varepsilon[\text{cl}(D_1)] \neq \emptyset$, since $I_i \setminus N_\varepsilon[\text{cl}(D_1)] \neq \emptyset$, and $f_\varepsilon[I_i] \cap f_\varepsilon[I_0 \setminus (D_1 \cup D_2)] = \emptyset$, for $i = 1, \dots, a$. Hence $f_\varepsilon[I_1]$ and $f_\varepsilon[I_2]$ both intersect $f_\varepsilon[\text{cl}(D_1)]$ and the component of $\eta_Y^{-1}[\text{cl}(\mathcal{C}) \setminus f_\varepsilon[\text{cl}(D_1)]]$ that contains V . For $i = 1, 2$, $f_\varepsilon[I_i] \cap f_\varepsilon[I_0 \setminus \text{cl}(D_1)] = \emptyset$, since $d[I_0 \setminus \text{cl}(D_1), I_i] \geq \varepsilon$. Therefore, $f_\varepsilon[I_1 \setminus \text{cl}(D_1)]$ and $f_\varepsilon[I_2 \setminus \text{cl}(D_1)]$ both intersect the component of $\eta_Y^{-1}[\text{cl}(\mathcal{C}) \setminus f_\varepsilon[\text{cl}(D_1)]]$ that contains V and intersect each other. But $d[I_1 \setminus D_1, I_2 \setminus D_2] \geq \varepsilon$, a contradiction. It follows that f_D does not map two points of order greater than 2 into the same point.

The case where $f_D^{-1}[T]$ contains one point of order greater than 2 and one point of order 1 and the case where $f_D^{-1}[T]$ contains a simple closed curve are proved with arguments similar to those of the proof above and are omitted. Since $f_D^{-1}[T]$ contains no simple closed curve and contains at most one point not of order 2, it follows from [3, Theorem 2, p. 142] that f_D is a near homeomorphism.

Corollary 5 [3, Theorem 1, p. 141]. *If X is a graph and $f: X \rightarrow Y$ is a refinable map, then Y is a graph, X and Y are homeomorphic, and f is a near homeomorphism.*

Proof. The corollary follows from Theorem 4 and the observation that Y is locally connected and therefore X , Y , and f are essentially the same as X_D , Y_D , and f_D , respectively.

2. FOCUS ON THE RANGE

Now we switch our attention to the range Y of a refinable map, give it some structure, and investigate what implications this has for the domain X . The main result (Theorem 9) shows that X is a θ_n -continuum if Y is a θ'_n -continuum, but a simple example is given to show that X need not be a θ'_n -continuum.

Theorem 6. *If Y is a θ_n -continuum, X is Y -like and has the property that $K(H) \setminus H$ has void interior for each subcontinuum H of X , then X is a θ'_n -continuum.*

Proof. Because of the hypothesis concerning the set function K it suffices to show that X is a θ_n -continuum. Suppose that X is not a θ_n -continuum. Let H be a subcontinuum of X such that $X \setminus H = \bigcup_{i=1}^{n+1} A_i$ where $\{A_i | i = 1, \dots, n+1\}$ is a collection of mutually separated sets. Since $K(H) \setminus H$ has void interior, there exists a subcontinuum R such that $H \subseteq \text{Int}(R) \subseteq R$ and $A_i^* = A_i \setminus R \neq \emptyset$ for $i = 1, \dots, n+1$. Observe that $\{\text{cl}(A_i^*) | i = 1, \dots, n+1\}$ is a collection of nonempty mutually separated closed sets. For each $i = 1, \dots, n+1$, choose $x_i \in A_i^*$ and let $\varepsilon_i = \min\{d(x_i, R) | i = 1, \dots, n+1\}$. Let $\varepsilon_2 = \min\{d(\text{cl}(A_i^*), \text{cl}(A_j^*)) | 1 \leq i < j \leq n+1\}$, and take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$.

Now let $f: X \rightarrow Y$ be an ε -map and note that $f(R)$ is a continuum with $Y \setminus f(R) \neq \emptyset$. Since Y is a θ_n -continuum, $Y \setminus f(R) = \bigcup_{i=1}^j B_i$ where $j \leq n$ and B_k is a connected set for $k = 1, \dots, j$. Since $x_i \in A_i^*$ and $d(x_i, R) \geq \varepsilon$, $f(A_i^*) \cap [\bigcup_{i=1}^j B_i] \neq \emptyset$ for $i = 1, \dots, n+1$. Because $j < n+1$, it follows that there exists an integer j_0 such that B_{j_0} intersects at least two members of the collection $\{f(\text{cl}(A_i^*)) | i = 1, \dots, n+1\}$. But this collection consists of mutually separated sets since $\varepsilon \leq \varepsilon_2$ and f is an ε -map. Thus the connected set B_{j_0} can be written as the union of a finite number (greater than 1) of mutually separated sets, a contradiction.

Corollary 7. *If Y is a θ_n -continuum, X has the property that $K(H) \setminus H$ has void interior for each subcontinuum H of X , and $f: X \rightarrow Y$ is a refinable map, then X is a θ'_n -continuum and so is Y .*

Proof. Since f is a refinable map, X is Y -like. Hence, by Theorem 6, X is a θ'_n -continuum, and by Theorem 3, Y is a θ'_n -continuum.

Corollary 8 [3, Corollary 2, p. 145]. *If Y is a graph, $f: X \rightarrow Y$ is a refinable map, and X is locally connected, then X is a graph, X and Y are homeomorphic, and f is a near homeomorphism.*

Proof. Since a graph is a θ_n -continuum for some n and any locally connected continuum has the property concerning the set function K in the last corollary, it follows from that corollary that X is a graph (a locally connected θ_n -continuum). Then by Corollary 5, X and Y are homeomorphic, and f is a near homeomorphism.

There exists a refinable map from the $\sin \frac{1}{x}$ continuum onto an interval Y . This shows that the locally connected condition in Corollary 8 cannot be replaced by the property of the set function K in Corollary 7, even when Y is very nice. Theorem 3 shows that the property of being a θ'_n -continuum held by the domain is preserved by a refinable map. If we reverse the role of the domain and range, we can show that the domain is a θ_n -continuum (but not necessarily a θ'_n -continuum) if the range is a θ'_n -continuum. The next theorem and example show these facts.

Theorem 9. *If Y is a θ'_n -continuum and $f: X \rightarrow Y$ is a refinable map, then X is a θ_n -continuum.*

Proof. Let Y_D be the canonical decomposition of Y and let η_Y be the quotient map from Y onto Y_D . Since $\eta_Y \circ f$ is monotone (established in the first paragraph of the proof of Theorem 4), $f^{-1}[\eta_Y^{-1}[L]]$ is a subcontinuum of X if L is a subcontinuum of Y_D . Such subcontinua of X cannot separate X into more than n components, since Y is a θ_n -continuum. To see this, assume that L is a subcontinuum of Y_D and $X \setminus f^{-1}[\eta_Y^{-1}[L]]$ has more than n components. Then the images under f of some two of them, C_1 and C_2 , must lie in the same component of $Y \setminus \eta_Y^{-1}[L]$, since the latter set has at most n components. So there exists a continuum L_1 in $Y_D \setminus L$ such that $\eta_Y^{-1}[L_1] \cap f[C_1]$ and $\eta_Y^{-1}[L_1] \cap f[C_2]$ are nonvoid. Hence $f^{-1}[\eta_Y^{-1}[L_1]]$ is not connected and the monotonicity of $\eta_Y \circ f$ is contradicted.

Let K be a subcontinuum of X and let $B = \{y_D \in Y_D \mid K \text{ intersects, but does not contain, } f^{-1}[\eta_Y^{-1}(y_D)]\}$. We wish to show that B is finite. Since Y_D contains only a finite number of points of order different from 2, it is sufficient to show that B contains only a finite number of points of order 2. The graph Y_D is the union of a finite collection $\{A_1, \dots, A_k\}$ of arcs whose non-end points are of order 2 in Y_D . Hence, it is sufficient to show that none of these arcs contains 3 points of B that are not end points of the arc. Suppose one of the arcs, without loss of generality A_1 , contains 3 such points a , b , and c where b separates A_1 between a and c . Since $\eta_Y[f[K]]$ is connected and $\{a, b, c\} \subseteq \eta_Y[f[K]]$, without loss of generality there is a point p of $\eta_Y[f[K]]$ between a and b in A_1 . Let a' , p'' , b' , and c'' be points of $f^{-1}[\eta_Y^{-1}(a)] \setminus K$, $f^{-1}[\eta_Y^{-1}(p)] \cap K$, $f^{-1}[\eta_Y^{-1}(b)] \setminus K$, and $f^{-1}[\eta_Y^{-1}(c)] \cap K$, respectively.

If f_ε is an ε -refinement of f , where $\varepsilon < d[\{a', b'\}, K]$ and is small enough otherwise, then $f_\varepsilon[\{a', b'\}] \cap f_\varepsilon[K] = \emptyset$ and the points $\eta_Y(f_\varepsilon(a'))$, $\eta_Y(f_\varepsilon(p''))$, $\eta_Y(f_\varepsilon(b'))$, and $\eta_Y(f_\varepsilon(c''))$ lie along $\text{Int}(A_1)$ in the same order as a , p , b , and c . But this is impossible because of the irreducibility properties of $\eta_Y^{-1}[A_1]$, since $f_\varepsilon[K]$ is a continuum. Therefore, B is a finite set.

Suppose X is not a θ_n -continuum. Then there is a subcontinuum K of X such that $X \setminus K$ has at least $n+1$ components. But $\eta_Y[f[K]]$ is a continuum in

Y_D so $X \setminus f^{-1}[\eta_Y^{-1}[\eta_Y[f[K]]]]$ has at most n components. Hence, some component C of $X \setminus K$ is contained in $f^{-1}[\eta_Y^{-1}[\eta_Y[f[K]]]]$. Since B is finite and C is connected, there is a y_D and Y_D such that $f^{-1}[\eta_Y^{-1}(y_D)]$ contains C . Let m be the order of y_D in Y_D , and let $[a_1, y_D], \dots, [a_m, y_D]$ be disjoint (except for y_D) arcs in Y_D such that, for $i = 1, \dots, m$, $[a_i, y_D] = [a_i, y_D] \setminus \{y_D\}$ contains only points of order 2. For $i = 1, \dots, m$, there is a point b_i in the open arc (a_i, y_D) such that either K contains $f^{-1}[\eta_Y^{-1}[[b_i, y_D]]]$ or $K \cap f^{-1}[\eta_Y^{-1}[[b_i, y_D]]] = \emptyset$. This follows from the finiteness of B and the fact that $\text{cl}(\eta_Y^{-1}[(a_i, y_D)])$ is an irreducible subcontinuum of a θ'_n -continuum. For $i = 1, \dots, m$, if K does not intersect $f^{-1}[\eta_Y^{-1}[[b_i, y_D]]]$, let K_i be the closure of the component of $f^{-1}[\eta_Y^{-1}[[b_i, y_D]]] \setminus K$ that contains $f^{-1}[\eta_Y^{-1}[[b_i, y_D]]]$. On the other hand, if K contains $f^{-1}[\eta_Y^{-1}[[b_i, y_D]]]$ let K_i be the component of $K \cap f^{-1}[\eta_Y^{-1}[[b_i, y_D]]]$ that contains $f^{-1}[\eta_Y^{-1}[[b_i, y_D]]]$.

Clearly, $\bigcup_{i=1}^m K_i \cup K$ is a continuum that does not intersect C . We now show that this leads to a contradiction. Let $x \in C$. Note that $f[\bigcup_{i=1}^m K_i \cup K]$ contains the neighborhood $\bigcup_{i=1}^m \eta_Y^{-1}[[b_i, y_D]]$ of $\eta_Y^{-1}(y_D)$, since $f[\bigcup_{i=1}^m K_i \cup K]$ is closed and $\eta_Y^{-1}(y_D)$ has void interior. If $\varepsilon < d(x, \bigcup_{i=1}^m K_i \cup K)$ and $N_\varepsilon[\eta_Y^{-1}(b_i)] \subseteq \eta_Y^{-1}[(a_i, y_D)]$, for $i = 1, \dots, m$, then any ε -refinement of f maps $\bigcup_{i=1}^m K_i \cup K$ onto a connected set that must be a neighborhood of $\eta_Y^{-1}(y_D)$ and maps x into that neighborhood, a contradiction. Since the assumption that K separates X into more than n components leads to a contradiction, X is a θ_n -continuum.

Example 10. A θ'_2 -continuum Y , a θ_2 -continuum X that is not a θ'_2 -continuum and a refinable map $f: X \rightarrow Y$.

Let X be the union of Knaster's planar indecomposable " S " continuum T with two end points p, q and an arc $[q, r]$ such that $T \cap [q, r] = \{q\}$. Let $Y = [0, 1]$ and let $f: X \rightarrow Y$ be a map that sends T into 0 and $[q, r]$ homeomorphically onto $[0, 1]$. If $0 < \varepsilon < 1$ let C_ε be an ε -chain from p to q in T . Define f_ε such that $f_\varepsilon(q) = \frac{\varepsilon}{2}$, f_ε maps $[q, r]$ homeomorphically onto $[\frac{\varepsilon}{2}, 1]$ in such a way that $d(f_\varepsilon(t), f(t)) < \varepsilon$ for $t \in [q, r]$, and f_ε maps T onto $[0, \frac{\varepsilon}{2}]$ by "following" the chain C_ε from p to q . Clearly f_ε is an ε -refinement of f . Thus f is refinable.

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