GENERIC FAMILIES AND MODELS OF SET THEORY WITH THE AXIOM OF CHOICE

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ABSTRACT. Let M be a countable transitive model of ZFC and A be a countable M -generic family of Cohen reals. We prove that there is no smallest transitive model N of ZFC that either $M \cup A \subseteq N$ or $M \cup \{A\} \subseteq N$. It is also proved that there is no smallest transitive model N of ZFC^- (ZFC theory without the power set axiom) such that $M \cup \{A\} \subseteq N$. It is also proved that certain classes of extensions of M obtained by Cohen generic reals have no minimal model.

0. INTRODUCTION

In the paper [1] Blass proved that for a given countable transitive model M of ZF and a countable M -generic family A of Cohen reals there exists no smallest transitive model N of ZF such that $M \cup A \subseteq N$. On the other hand Cohen's model $M[A]$ is the smallest transitive model N of ZF such that $M \cup \{A\} \subseteq N$. Similar questions can be asked about models of the theory ZF^- (Zermelo–Fraenkel set theory without the power set axiom and with the collection and comprehension schemes instead of the replacement scheme). Then $M[A]$ is also the smallest transitive model N of ZF^- such that $M \cup \{A\} \subseteq N$. It is an observation of Szczepaniak that the model $M(A) = \bigcup \{M[a] : a \subseteq A \text{ and } |a| < \omega\}$ is a model of ZF^- (see [5], for a proof see also [1] or [3]) and it is obvious that $M(A)$ is the smallest model N of ZF^- such that $M \cup A \subseteq N$.

In this paper we ask four analogous questions: If M is a countable transitive model of the theory ZFC (or ZFC^-) and A is a countable M -generic family of Cohen reals, does there exist the smallest model N of ZFC (or ZFC^- , respectively) such that $M \cup \{A\} \subseteq N$ or $M \cup A \subseteq N$? Since $M(A)$ satisfies the axiom of choice it is the smallest transitive model N of ZFC^- such that $M \cup A \subseteq N$. We will show that in the remaining three cases the answers are negative: the respective smallest models do not exist.

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We conclude the paper with investigating extensions of M obtained by adding many Cohen generic reals. We show that among such models there exists no minimal model containing a given generic family as an element.

1. NONEXISTENCE OF THE SMALLEST MODELS

We begin with introducing several notions of forcing. For any sets A and B we define $\mathbf{P}(A, B)$ as the set of functions $p: \text{dom}(p) \rightarrow B$, where $\text{dom}(p)$ is a finite subset of A , ordered by reverse inclusion. Next

$$\mathbf{P}(A) = \mathbf{P}(A, 2),$$

$$\mathbf{P} = \mathbf{P}(\omega),$$

$$\mathbf{Q}_\alpha = \mathbf{P}(\alpha \times \omega) \quad \text{for } \alpha \in On,$$

$$\mathbf{Q} = \mathbf{Q}_{\omega_1^M} \quad \text{where } M \text{ is a fixed countable transitive model of } ZFC.$$

For any M -generic filter $G \subseteq \mathbf{Q}_\alpha$ we define M -generic filters $(G)_\xi \subseteq \mathbf{P}$ for $\xi \in \alpha$ with the formula

$$p \in (G)_\xi \text{ iff } \exists q \in G \forall n \in \text{dom}(p) [\langle \xi, n \rangle \in \text{dom}(q) \& q(\xi, n) = p(n)].$$

Next we define

$$\tilde{G} = \{(G)_\xi : \xi \in \alpha\}.$$

Definition 1.1. A family $\mathcal{G}$ of filters in $\mathbf{P}$ is an M -generic family iff for any finite number of filters $G_1, \dots, G_n \in \mathcal{G}$ the n -tuple $\langle G_1, \dots, G_n \rangle$ is M -generic over $\mathbf{P}^n$ and for any forcing condition $p \in \mathbf{P}$ there exists $G \in \mathcal{G}$ such that $p \in G$.

It is well known that for any M -generic filter $G \subseteq \mathbf{Q}_\alpha$ the family $\tilde{G}$ is M -generic. For countable generic families the converse is also true (see [2]).

Lemma 1.2 (Friedman, Cohen). *If A is a countable M -generic family then there exists an M -generic filter $G \subseteq \mathbf{Q}_\omega$ such that $\tilde{G} = A$. Moreover $M[A]$ is a model of $ZF + \neg AC$.*

We omit the proof of the lemma. $\square$

A well-known property of the model $M[A]$ is given by the following

Lemma 1.3. *If $x \subseteq M$ and $x \in M[A]$ then there exists a finite subset $a \subseteq A$ such that $x \in M[a]$.*

We omit the proof of the lemma. $\square$

In the model $M[A]$ we can define the following notion of forcing $\mathbf{R}_\alpha$ for $\alpha \in On - \omega$:

$$\begin{aligned} \mathbf{R}_\alpha &= \{p \in \mathbf{P}(\alpha, A) : p \text{ is one-to-one}\}, \\ p \leq q &\quad \text{iff } q \subseteq p \quad \text{for } p, q \in \mathbf{R}_\alpha. \end{aligned}$$

Now we can prove the following

Lemma 1.4. *If A is a countable M -generic family and $\alpha \in (On \cap M) - \omega$ then for any $M[A]$ -generic filter $H \subseteq \mathbb{R}_\alpha$ there exists an M -generic filter $G \subseteq \mathbb{Q}_\alpha$ such that $\tilde{G} = A$ and $M[A][H] = M[G]$.*

Proof. Let $H \subseteq \mathbb{R}_\alpha$ be an $M[A]$ -generic filter. We identify generic filters with generic functions; under this identification H is a one-to-one function from α onto A .

We define a filter G in $\mathbb{Q}_\alpha$ by the formula

$$G(\xi, n) = H(\xi)(n).$$

Then obviously $\tilde{G} = A$ and $M[A][H] = M[G]$. We must only show that the filter G is M -generic. Let $D \in M$ be a dense subset of $\mathbb{Q}_\alpha$.

We define a subset E of $\mathbb{R}_\alpha$:

$$p \in E \text{ iff there exists } q \in D \text{ such that for any } \xi \in \alpha \text{ and } n \in \omega \\ \text{if } \langle \xi, n \rangle \in \text{dom}(q) \text{ then } \xi \in \text{dom}(p) \text{ and } p(\xi)(n) = q(\xi, n).$$

Obviously $E \in M[A]$. We will show that the set E is dense in $\mathbb{R}_\alpha$. This will suffice because then $E \cap H \neq \emptyset$ and for $p \in E \cap H$ and $q \in D$ such as in the definition of E we have $q \in G$, thus $D \cap G \neq \emptyset$.

In order to show that E is dense in $\mathbb{R}_\alpha$ we take an arbitrary condition $p_0 \in \mathbb{R}_\alpha$.

$$\text{Let } a = \text{dom}(p_0) = \{\xi_1, \dots, \xi_n\} \subseteq \alpha,$$

$$p_0(\xi_i) = G_i \in A \quad \text{for } i = 1, \dots, n.$$

We define a set $D_a \subseteq \mathbb{P}(a \times \omega)$ as follows:

$$D_a = \{q \in \mathbb{P}(a \times \omega) : \exists p \in D (q = p \upharpoonright a \times \omega)\}.$$

Obviously, $D_a \in M$ and D_a is a dense subset of $\mathbb{P}(a \times \omega)$. Since the tuple $\langle G_1, \dots, G_n \rangle$ is M -generic in $\mathbb{P}^n$, there exists $q \in D_a$ such that $\langle G_1, \dots, G_n \rangle$ extends q (under a suitable identification of the notions of forcing $\mathbb{P}^n$ and $\mathbb{P}(a \times \omega)$). Let $p \in D$ be such that $q = p \upharpoonright a \times \omega$.

Suppose that $\text{dom}(p) \subseteq (a \cup b) \times \omega$ and $a \cap b = \emptyset$. Let $b = \{\zeta_1, \dots, \zeta_m\}$.

We find distinct filters $G'_1, \dots, G'_m \in A - \{G_1, \dots, G_n\}$ such that the tuple $\langle G'_1, \dots, G'_m \rangle$ extends $p \upharpoonright b \times \omega$. Then we define a condition $r \in \mathbb{R}_\alpha$:

$$\text{dom}(r) = a \cup b,$$

$$r(\xi_i) = G_i,$$

$$r(\zeta_i) = G'_i.$$

Then obviously $r \leq p_0$ and $r \in E$. $\square$

The following is a special case of a corollary to the product lemma.

Lemma 1.5. *If a pair of filters $\langle H_1, H_2 \rangle$ is $M[A]$ -generic over $\mathbb{R}_\alpha^2$ then $M[A][H_1] \cap M[A][H_2] = M[A]$.*

For a proof see [4]. $\square$

Corollary 1.6. *For any M -generic family A and any infinite $\alpha \in \text{On} \cap M$ there exist M -generic filters $G_1, G_2 \subseteq \mathbb{Q}_\alpha$ such that $\tilde{G}_1 = \tilde{G}_2 = A$ and $M[G_1] \cap M[G_2] = M[A]$.*

Proof. We take an $M[A]$ -generic pair of filters $\langle H_1, H_2 \rangle$ over $\mathbb{R}_\alpha^2$ and find M -generic filters $G_1, G_2 \subseteq \mathbb{Q}_\alpha$ such that $M[A][H_1] = M[G_1]$ and $M[A][H_2] = M[G_2]$ and $\tilde{G}_1 = \tilde{G}_2 = A$.

Then by Lemma 1.5

$$M[G_1] \cap M[G_2] = M[A][H_1] \cap M[A][H_2] = M[A]. \quad \square$$

Now we can prove the following:

Theorem 1.7. *For any countable M -generic family A there exists no smallest model N of ZFC such that $M \cup A \subseteq N$.*

Proof. We take M -generic filters $G_1, G_2 \subseteq \mathbb{Q}_\omega$ such that $M[G_1] \cap M[G_2] = M[A]$ and $\tilde{G}_1 = \tilde{G}_2 = A$. Now it is enough to show that no submodel N of $M[A]$ such that $M \cup A \subseteq N$ can be model of ZFC . Let us assume that N is such a model. Then obviously $P(\omega) \cap N = P(\omega) \cap M[A]$. In N there exists a function $f: \alpha \times \omega \rightarrow 2$ such that $\alpha = |P(\omega) \cap N|^N$ and

$$P(\omega) \cap N = \{ \{n \in \omega: f(\xi, n) = 0\} : \xi \in \alpha \}.$$

Then there exists a finite set $a \subseteq A$ such that $f \in M[a]$, which gives a contradiction because $A \in M[f]$. $\square$

We also have

Theorem 1.8. *For any countable M -generic family A there exists no smallest model N of ZFC such that $M \cup \{A\} \subseteq N$.*

Proof. We take M -generic filters $G_1, G_2 \subseteq \mathbb{Q}_\omega$ such that $M[G_1] \cap M[G_2] = M[A]$ and $\tilde{G}_1 = \tilde{G}_2 = A$. If N is a smallest model of ZFC such that $M \cup \{A\} \subseteq N$ then $N \subseteq M[A]$. On the other hand, since $A \in N$, we have $M[A] \subseteq N$. This is a contradiction because $M[A] \models \neg AC$. $\square$

Finally, we have

Theorem 1.9. *For any countable M -generic family A there exist no smallest model N of ZFC^- such that $M \cup \{A\} \subseteq N$.*

Proof. We recall that if $G \subseteq \mathbb{Q}$ is an M -generic filter then the model $M_G^* = \bigcup \{M[x, \tilde{G}]: x \in P(\omega) \cap M[G]\}$ is a model of ZFC^- (for a proof see [3]). Now we take M -generic filters $G_1, G_2 \subseteq \mathbb{Q}$ such that $\tilde{G}_1 = \tilde{G}_2 = A$ and $M[G_1] \cap M[G_2] = M[A]$. Then $M_{G_1}^* \cap M_{G_2}^* \subseteq M[A]$. On the other hand if N is a model of ZFC^- such that $M \cup \{A\} \subseteq N$ then $M[A] \subseteq N$. Therefore $M_{G_1}^* \cap M_{G_2}^* = M[A]$. This gives a contradiction because the smallest model N of ZFC^- such that $M \cup \{A\} \subseteq N$ would have to be equal to $M[A]$ and $M[A] \models \neg AC$. $\square$

We conclude this section with the observation that a lemma converse to Lemma 1.4 is also true.

Lemma 1.10. *If A is a countable M -generic family and $\alpha \in \text{On} \cap M$ is infinite then for any M -generic filter $G \subseteq \mathbb{Q}_\alpha$ such that $\tilde{G} = A$ there exists an $M[A]$ -generic filter $H \subseteq \mathbb{R}_\alpha$ such that $M[A][H] = M[G]$.*

Proof. The filter G defines the following function $g: \alpha \xrightarrow{1-1} A$

$$g(\xi)(n) = \left(\bigcup G \right)(\xi, n) \quad \text{for } \xi \in \alpha \text{ and } n \in \omega.$$

The filter $H \subseteq \mathbb{R}_\alpha$ is defined as

$$H = \{g \upharpoonright a: a \subseteq \alpha \text{ \& } |a| < \omega\} = \{q \in \mathbb{R}_\alpha: q \subseteq g\}.$$

It is obvious that $M[A][H] = M[G]$. We will show that H is an $M[A]$ -generic filter in $\mathbb{R}_\alpha$. The filter H has a canonical name $\dot{H}$ with the following property:

If $\dot{a}_i$ is a canonical name for $g(\xi_i)$, $i = 1, \dots, n$, then

$$1 \Vdash \{\langle \xi_i, \dot{a}_i \rangle: i = 1, \dots, n\} \in \dot{H}.$$

Now let $D \in M[A]$ be a dense subset of $\mathbb{R}_\alpha$. We take a name $\dot{D}$ for D such that there exists a subset $X = \{\xi_1, \dots, \xi_n\} \subseteq \alpha$ such that for any permutation $\pi: \alpha \rightarrow \alpha$ such that $\pi \in M$ and $\pi(\xi_i) = \xi_i$ for $i = 1, \dots, n$ we have $\pi(\dot{D}) = \dot{D}$. Now we take a condition $p_0 \in G$ such that $p_0 \Vdash \dot{D}$ is dense in $\dot{\mathbb{R}}_\alpha$ and define

$$E = \{p \in \mathbb{Q}_\alpha: p \Vdash \dot{D} \cap \dot{H} \neq \emptyset\}.$$

It is enough to show that E is dense in $\mathbb{Q}_\alpha$ below p_0 . Let $p \in \mathbb{Q}_\alpha$ and $p \leq p_0$. Let $\text{dom}(p) \subseteq Z \times \omega$, $Z = \{\zeta_1, \dots, \zeta_m\}$ and $X \subseteq Z$. We take a condition

$$q = \{\langle \zeta_i, g(\zeta_i) \rangle: i = 1, \dots, m\} \in \mathbb{R}_\alpha.$$

Obviously $p \Vdash \dot{q} \in \dot{\mathbb{R}}_\alpha$.

We take $p_1 \leq p$ and $q_1 \leq q$ such that $p_1 \Vdash \dot{q}_1 \in \dot{D}$. We assume that

$$q_1 = q \cup \{\langle \eta_j, g(\rho_j) \rangle: j = 1, \dots, k\}$$

and $\zeta_i \neq \eta_j$, $\zeta_i \neq \rho_j$ for $i = 1, \dots, m$ and $j = 1, \dots, k$.

We define a permutation $\pi: \alpha \rightarrow \alpha$ so that

$$\begin{aligned} \pi(\rho_j) &= \eta_j \quad \text{for } j = 1, \dots, k, \\ \pi(\zeta_i) &= \zeta_i \quad \text{for } i = 1, \dots, m. \end{aligned}$$

Then

$$\pi p_1 \Vdash \pi \dot{q}_1 \in \dot{D}.$$

But $\pi p_1 \leq p$ and $\pi \dot{D} = D$. Next

$$\pi q_1 = q \cup \{\langle \eta_i, g(\eta_i) \rangle: i = 1, \dots, k\}$$

and hence

$$\pi p_1 \Vdash \pi \dot{q}_1 \in \dot{H}.$$

Therefore $\pi p_1 \Vdash \pi \dot{q}_1 \in \dot{D} \cap \dot{H}$ and finally $\pi p_1 \in E$. $\square$

2. NONEXISTENCE OF MINIMAL COHEN EXTENSIONS

We begin with the following:

Lemma 2.1. *For any M -generic filter $G \subseteq \mathbb{Q}_\omega$ there exists an M -generic filter $H \subseteq \mathbb{Q}_\omega$ such that $\tilde{G} = \tilde{H}$ and*

$$P(\omega) \cap M[H] \subsetneq P(\omega) \cap M[G].$$

Proof. Let $\mathbf{P}_1$ be the following dense subset of $\mathbb{Q}_\omega$:

$$p \in \mathbf{P}_1 \equiv \exists n, m \in \omega (\text{dom}(p) = n \times m)$$

and let $\mathbf{P}_2$ be the following dense subset of $\mathbb{Q}_\omega \times \mathbb{Q}_\omega \times \mathbf{P}$:

$$\langle q, r, s \rangle \in \mathbf{P}_2 \equiv \exists n_1, n_2, m \in \omega (\text{dom}(q) = n_1 \times m$$

$$\& \text{dom}(r) = n_2 \times m \& \text{dom}(s) = n_1 + n_2 \& |s^{-1}(0)| = n_1 \& |s^{-1}(1)| = n_2).$$

Now we define an isomorphism $h: \mathbf{P}_1 \rightarrow \mathbf{P}_2$. For $p \in \mathbf{P}_1$ such that $\text{dom}(p) = n \times m$ we put $h(p) = \langle q, r, s \rangle$, where:

$$\begin{aligned} \text{dom}(s) &= n, \\ s(i) &= p(i, 0) \quad \text{for } i < n, \\ n_1 &= |s^{-1}(0)|, \\ n_2 &= |s^{-1}(1)| \\ \text{dom}(q) &= n_1 \times (m - 1), \\ \text{dom}(r) &= n_2 \times (m - 1), \end{aligned}$$

if $s(i) = 0$ and $|\{j < i : s(j) = 0\}| = k$ then

$$q(k, l) = p(i, l + 1),$$

if $s(i) = 1$ and $|\{j < i : s(j) = 1\}| = k$ then

$$r(k, l) = p(i, l + 1).$$

The definition of h is best illustrated by an example. Let a condition p be represented by the following matrix

$$\begin{array}{|c|c|c|c|c|c|c|c|} \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline a & b & c & d & e & f & g & h \\ \hline 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ \hline \end{array}$$

Then the following matrices represent conditions q, r, s such that $h(p) = \langle q, r, s \rangle$:

$$\begin{array}{l} q: \begin{array}{|c|c|c|c|c|} \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline a & b & e & g & h \\ \hline \end{array} \\ r: \begin{array}{|c|c|c|} \hline \vdots & \vdots & \vdots \\ \hline c & d & f \\ \hline \end{array} \\ s: \begin{array}{|c|c|c|c|c|c|c|c|} \hline 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ \hline \end{array} \end{array}$$

Now let $K = h(G \cap \mathbf{P}_1)$. Then K is an M -generic filter in $\mathbf{P}_2$ which generates in a natural way an M -generic triple of filters $K_1 \times K_2 \times K_3 \subseteq \mathbf{Q}_\omega \times \mathbf{Q}_\omega \times \mathbf{P}$. Now we split the filter K_3 into two M -generic filters H_3 and H_4 :

$$\begin{aligned} p \in H_3 &\equiv \exists q \in K_3 \forall n \in \text{dom}(p) (2n \in \text{dom}(q) \& p(n) = q(2n)), \\ p \in H_4 &\equiv \exists q \in K_3 \forall n \in \text{dom}(p) (2n+1 \in \text{dom}(q) \& p(n) = q(2n+1)). \end{aligned}$$

We also put $H_1 = K_1$ and $H_2 = K_2$. Finally

$$H = h^{-1}(H_1 \times H_2 \times H_3 \cap \mathbf{P}_2).$$

Then obviously $M[H] = M[H_1, H_2, H_3] \subseteq M[G]$. Since $H_4 \in M[G]$ and H_4 is $M[H]$ -generic in $\mathbf{P}$ we have

$$P(\omega) \cap M[H] \subsetneq P(\omega) \cap M[G].$$

It remains to show that $\tilde{G} = \tilde{H}$. For any M -generic filter $F \subseteq \mathbf{P}$ we define a filter $\bar{F}$ with the formula

$$\bar{F}(n) = F(n+1)$$

(we recall that an M -generic filter is identified with the corresponding M -generic function). Now we observe that

$$\begin{aligned} F \in \tilde{G} &\equiv (F(0) = 0 \& \bar{F} \in \tilde{K}_1) \vee (F(0) = 1 \& \bar{F} \in \tilde{K}_2) \\ &\equiv (F(0) = 0 \& \bar{F} \in \tilde{H}_1) \vee (F(0) = 1 \& \bar{F} \in \tilde{H}_2) \\ &\equiv F \in \tilde{H}. \quad \square \end{aligned}$$

Corollary 2.2. *For any M -generic family A the class of models*

$$\{M[G]: G \subseteq \mathbf{Q}_\omega \text{ is } M\text{-generic} \& \tilde{G} = A\}$$

has no minimal element. $\square$

Now it is easy to generalize Lemma 2.1 to the case of M -generic filters in $\mathbf{Q}$. Namely

$$\mathbf{Q} \cong \mathbf{P}(\omega \times \omega) \times \mathbf{P}((\omega_1^M - \omega) \times \omega).$$

Any M -generic filter $G \subseteq \mathbf{Q}$ decomposed as $G_1 \times G_2$ where

$$\begin{aligned} G_1 &= G \cap \mathbf{P}(\omega \times \omega), \\ G_2 &= G \cap \mathbf{P}((\omega_1^M - \omega) \times \omega) \end{aligned}$$

and

$$G_1 \text{ is } M[G_2]\text{-generic}.$$

From Lemma 2.1 it follows that there exists an $M[G_2]$ -generic filter $H_1 \subseteq \mathbf{P}(\omega \times \omega)$ such that $\tilde{H}_1 = \tilde{G}_1$ and

$$P(\omega) \cap M[G_2][H_1] \subsetneq P(\omega) \cap M[G_2][G_1].$$

We put $H = H_1 \times G_2$ and observe that

$$P(\omega) \cap M[H] \subsetneq P(\omega) \cap M[G]$$

and

$$\tilde{G} = \tilde{G}_1 \cup \tilde{G}_2 = \tilde{H}_1 \cup \tilde{G}_2 = \tilde{H}. \quad \square$$

This obviously gives the following

Corollary 2.3. *For any countable M -generic family A the class of models*

$$\{M_G^*: G \subseteq \mathbb{Q} \text{ is } M\text{-generic} \ \& \ \tilde{G} = A\}$$

has no minimal element. $\square$

We conclude the paper with the following

Problem 2.4. Does there exist a minimal model N of ZFC (or ZFC^-) such that $M \cup \{A\} \subseteq N$? Does there exist a minimal model of N of ZFC such that $M \cup A \subseteq N$?

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