UNCONDITIONAL BASES IN COUNTABLY- $\mathcal{L}_1$ SPACES

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ABSTRACT. In this paper, we show that a countably- $\mathcal{L}_1$ space which has an unconditional basis is isomorphic to some echelon sequence space of order 1. As a consequence, a countably- $\mathcal{L}_1$ space with a basis is nuclear if all its bases are unconditional (this gives a partial answer to a conjecture of Wojtynski). We also study those countably- $\mathcal{L}_1$ spaces on which a Fréchet lattice structure can be defined.

INTRODUCTION

In [9], it was assumed by Wojtynski that a Fréchet space E with a basis is nuclear provided each basis in E is unconditional. He also proved that the conjecture holds when E is Hilbertisable (note that the conjecture is already established for Banach spaces since in every infinite-dimensional Banach space with a basis there exists a conditional basis; see Pelczynski and Singer, *Studia Math.* **25** (1964), 5–25). Here, we provide a positive new approach to the Wojtynski conjecture. Indeed, we show that a countably- $\mathcal{L}_1$ space with a basis is nuclear if each basis in E is unconditional. To this end, we prove that a countably- $\mathcal{L}_1$ space which possesses an unconditional basis is isomorphic to some echelon sequence space of order 1, $\lambda(a_n^k)$ (this is a generalization, to the framework of Fréchet spaces, of a well-known result of Lindenstrauss and Pelczynski; see [3, Theorem 6.1]).

In this paper, we also study those countably- $\mathcal{L}_1$ spaces which have a Fréchet lattice structure, and show that such spaces must be isomorphic to some echelon Köthe space $\Lambda(X, \beta, \mu, g_k)$. As an application, we give a new proof, in the framework of Fréchet spaces, of a known result of Komura and Koshi on nuclear vector lattices [2].

The paper consists of four sections. In the first section we provide some concepts, notations and preliminary results. Two useful lemmas (in fact, Lemma 2.2 is the key of this paper) concerning unconditional Schauder decompositions of Fréchet spaces are proved in §2. The third section is devoted to the main

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results (unconditional bases in countably- $\mathcal{L}_1$ spaces). Some other interesting applications of Lemma 2.2 are given in §4.

1. NOTATIONS AND PRELIMINARIES

The vector spaces we shall use are defined over the field $\mathbb{R}$ of reals. Standard references for notations and concepts are [1, 6 and 7].

Let E be a Fréchet space and let $(\|\cdot\|_k)_k$ be a fundamental system of seminorms (f.s.s.) (i.e., denoting $U_k = \{x \in E; \|x\|_k \leq 1\}$, then $(U_k)_k$ is a 0-neighborhood basis). We shall denote by E_k the local Banach space spanned by the seminorm $\|\cdot\|_k$, i.e. $(E/\|\cdot\|_k^{-1}(0), \|\cdot\|_k)^\wedge$, and by $I_k: E \rightarrow E_k$ the canonical mapping ($I_k(x) = \hat{x}$), $k \in \mathbb{N}$. The fact that $\|I_k(x)\|_k = \|x\|_k$, $x \in E$, $k \in \mathbb{N}$, will be used. E is said to be a *countably- $\mathcal{L}_1$ space* if it possesses a f.s.s. $(\|\cdot\|_k)_k$ such that E_k is an $\mathcal{L}_{1,\lambda_k}$ -space for some $\lambda_k \geq 1$, $k \in \mathbb{N}$ (see [3] for the definition of $\mathcal{L}_{1,\lambda}$ -spaces).

A Fréchet lattice is an ordered topological vector space which is Fréchet and it possesses a fundamental system $(\|\cdot\|_k)_k$ of monotonic seminorms (i.e., if $|x| \leq |y|$ then $\|x\|_k \leq \|y\|_k$, $\forall k \in \mathbb{N}$). Let E, F be Fréchet lattices and let $T: E \rightarrow F$ be a linear map (operator); T is said to be positive if $Tx \geq 0$ for all $x \geq 0$. It is easy to see that a positive operator between Fréchet lattices is continuous. An order isomorphism is a linear isomorphism $T: E \rightarrow F$ so that T and T^{-1} are positive. An element $a \in E$ is called an atom if given $x, y \in E$ such that $x \wedge y = 0$ and $x + y = |a|$, then either $x = a$ or $y = a$. E is called discrete if there exists a maximal orthogonal system $(x_i)_{i \in I}$ of E consisting of atoms.

Let (X, β, μ) be a measure space localizable and with the finite subset property [10]. A β -measurable set A of positive measure is called an atom whenever for any β -measurable subset B of A we have either $\mu(B) = 0$ or $\mu(A - B) = 0$. (X, β, μ) is called purely atomic if every β -measurable set A of finite measure is a union of atoms. The support of a β -measurable function $f: X \rightarrow \mathbb{R}$ is defined as $S(f) = \{x \in X; f(x) \neq 0\}$. Let $(g_k)_k$ be a sequence of β -measurable functions such that $0 \leq g_k(x) \leq g_{k+1}(x)$, μ -a.e. and $\mu(X - \bigcup_{k=1}^\infty S(g_k)) = 0$. Then, for every $p \in \mathbb{R}$, $p \geq 1$, the echelon Köthe space of order p is defined as the space of all (equivalence classes of) β -measurable functions such that

$$\|f\|_k = \left(\int_X |f|^p g_k d\mu \right)^{1/p} < \infty, \quad \forall k \in \mathbb{N}.$$

It will be denoted by $\Lambda^p(X, \beta, \mu, g_k)$ (or Λ^p to short) and it is a Fréchet lattice with the Fréchet topology induced by the seminorms $(\|\cdot\|_k)_k$ and the order defined as $f \leq g$ iff $f(x) \leq g(x)$ μ -a.e. If (X, β, μ) is σ -finite and purely atomic then Λ^p is isomorphic (moreover, it is order isomorphic) to an echelon sequence space $\lambda^p(a_n^k)$ (see [5]). The next result, that we need below, has been proved in [5].

Lemma A. Let $\Lambda^p(X, \beta, \mu, g_k)$ be a Montel echelon Köthe space. Then (X, β, μ) is σ -finite and purely atomic.

We shall also need the next representation theorem [4].

Theorem B. Let E be a Fréchet lattice and let $(\|\cdot\|_k)_k$ be a f.s.s. defining the topology of E , so that for some $p \geq 1$ and every $x, y \in E$, we get

$$\|x + y\|_k^p = \|x\|_k^p + \|y\|_k^p, \text{ if } x \wedge y = 0, \forall k \in \mathbb{N}.$$

Then there exists an echelon Köthe space $\Lambda^p(X, \beta, \mu, g_k)$ order isomorphic to E . (*)

2. UNCONDITIONAL SHAUDER DECOMPOSITIONS

Definition. Let E be a Fréchet space and let $(M_i)_{i \in I}$ be a set of subspaces of E . The set $(M_i)_i$ is said to be an unconditional Schauder decomposition of E if every $x \in E$ has a unique representation of the form $x = \sum_i x_i$, $x_i \in M_i$, $i \in I$, such that the induced projections $P_i: E \rightarrow M_i$, $P_i(x) = x_i$, are continuous and, in addition, the series $x = \sum_i x_i$ converges unconditionally for every $x \in E$.

The proof of the next lemma is similar to the proof of Theorem 1.19 in [8]. For the sake of completeness we include a sketch of the same.

Lemma 2.1. Let $(M_i)_{i \in I}$ be an unconditional Schauder decomposition of the Fréchet space E , and let $(\|\cdot\|_k)_k$ be a f.s.s. defining the topology of E . Then, for every $k \in \mathbb{N}$, there is an index $\sigma(k) \in \mathbb{N}$ such that

$$\left\| \sum_i \varepsilon_i x_i \right\|_k \leq \left\| \sum_i x_i \right\|_{\sigma(k)}$$

for every $x = \sum_i x_i \in E$, $\varepsilon_i = \pm 1$.

Proof. Let us denote $U_k = \{x \in E; \|x\|_k \leq 1\}$, $\Omega = \{F \subset I; \text{card}(F) < \infty\}$ and $d = \{(\varepsilon_i)_i; \varepsilon_i = \pm 1\}$, and consider

$$U'_k = \left\{ x = \sum_i x_i \in E; S_{F, \varepsilon}(x) = \sum_{i \in F} \varepsilon_i x_i \in U_k, \forall F \in \Omega, \varepsilon = (\varepsilon_i) \in d \right\}.$$

For each $x = \sum_i x_i \in E$, the series $\sum_i x_i$ is weakly unconditionally Cauchy, thus the net of operators $\{S_{F, \varepsilon}; F \in \Omega, \varepsilon \in d\}$ is pointwise bounded and consequently $(U'_k)_k$ is a 0-neighbourhood basis for a topology T_d [8, Lemma 1.7]. Now, the subnet $\{S_{F, 1}; F \in \Omega\}$ fulfills $S_{F, 1}(x) \rightarrow x$, for all $x \in E$, so T_d is stronger than the topology of E , say T . In addition, $S_{F, \varepsilon}$ is continuous for all $F \in \Omega$, $\varepsilon \in d$, and E is barreled, hence $T_d = T$. Then, given $k \in \mathbb{N}$, there is $\sigma(k) \in \mathbb{N}$ so that $U_{\sigma(k)} \subset U'_k$ and the result follows. $\square$

(*) In this paper we always work with $p = 1$ and we denote by $\Lambda(X, \beta, \mu, g_k)$ (or by Λ) the echelon Köthe spaces of order 1.

We need the following inequality [3, p. 295]:

Let $(u_i)_{i \in I}$ be a generalized sequence in an $\mathcal{L}_{1,\lambda}$ space such that $\sum_i u_i$ converges unconditionally. Then

$$\left(\sum_i \|u_i\|^2 \right)^{1/2} \leq \lambda K_G \sup_{\varepsilon_i = \pm 1} \left\| \sum_i \varepsilon_i u_i \right\|$$

where K_G is the Grothendieck constant.

In the next lemma, I denotes a finite or infinite index set. Certainly, the result is straightforward if I is finite, but we are interested in the inequality we shall obtain at the end of the proof.

Lemma 2.2. *Let E be a countably- $\mathcal{L}_1$ space, let $(\|\cdot\|_k)_k$ be a f.s.s. of E , and let $(M_i)_{i \in I}$ be an unconditional Schauder decomposition of E . Then E is isomorphic to*

$$1^1(M_i) = \left\{ (x_i) \in \prod_{i \in I} M_i; \|\|(x_i)\|\|_k = \sum_i \|x_i\|_k < \infty, \forall k \in \mathbb{N} \right\}$$

$(1^1(M_i))$ is supposed to be endowed with the Fréchet topology induced by the seminorms $\|\|\cdot\|\|_k$, $k \in \mathbb{N}$.

Proof. It is clear that $1^1(M_i)$ does not depend of the f.s.s. chosen to define the topology of E . So, we can assume that $(\|\cdot\|_k)_k$ is such that E_k is an $\mathcal{L}_{1,\lambda_k}$ space for some $\lambda_k \geq 1$ and all $k \in \mathbb{N}$.

Let us now denote by $P_k(\cdot)$ the gauge of U_k^0 in E' , where $U_k = \{x \in E; \|x\|_k \leq 1\}$, $k \in \mathbb{N}$. Let us fix $k \in \mathbb{N}$ and let $\sigma(k)$ be as in Lemma 2.1. Then, let us choose $x'_i \in M'_i$ such that $P_k(x'_i) = 1$, for all $i \in I$ (note that M'_i is a complemented subspace of E' , $i \in I$). Now, given $x \in E$, $x = \sum_i x_i$, we get

$$\begin{aligned} \left(\sum_i |x'_i(x_i)|^2 \right)^{1/2} &\leq \left(\sum_i \|x_i\|_k^2 \right)^{1/2} = \left(\sum_i \|I_k(x_i)\|_k^2 \right)^{1/2} \\ &\leq \lambda_k K_G \sup_{\varepsilon_i = \pm 1} \left\| \sum_i \varepsilon_i I_k(x_i) \right\|_k \\ &= \lambda_k K_G \sup_{\varepsilon_i = \pm 1} \left\| \sum_i \varepsilon_i x_i \right\|_k \leq \lambda_k K_G \left\| \sum_i x_i \right\|_{\sigma(k)}. \end{aligned}$$

Then, we can define a continuous operator $T: E \rightarrow 1^2(I)$, $\sum_i x_i \rightarrow (x'_i(x_i))$ such that $\|T(x)\| \leq \lambda_k K_G \|x\|_{\sigma(k)}$, so, there is a continuous operator $T_{\sigma(k)}: E_{\sigma(k)} \rightarrow 1^2(I)$ such that $T_{\sigma(k)} \cdot I_{\sigma(k)} = T$ and $\|T_{\sigma(k)}\| \leq \lambda_k K_G$. The operator $T_{\sigma(k)}$ is absolutely summing [3, Theorem 4.1], thus, there is a constant $a_1 = a_1(T_{\sigma(k)})$

such that

$$\begin{aligned} \sum_i |x'_i(x_i)| &= \sum_i \|Tx_i\| = \sum_i \|T_{\sigma(k)} \cdot I_{\sigma(k)}(x_i)\| \\ &\leq a_1 \sup \left\{ \sum_i |x'(I_{\sigma(k)}x_i)|; P_{\sigma(k)}(x') = 1 \right\} \\ &\leq a_1 \sup \left\{ \left\| \sum_i \varepsilon_i I_{\sigma(k)}(x_i) \right\|_{\sigma(k)}; \varepsilon_i = \pm 1 \right\} \\ &= a_1 \sup \left\{ \left\| \sum_i \varepsilon_i x_i \right\|_{\sigma(k)}; \varepsilon_i = \pm 1 \right\} \leq a_1 \|x\|_{\sigma(\sigma(k))}. \end{aligned}$$

It is known that $a_1 \leq \lambda_{\sigma(k)} K_G \|T_{\sigma(k)}\|$ [3, p. 286]. Thus

$$\sum_i \|x_i\|_k = \sup \left\{ \sum_i |x'_i(x_i)|; P_k(x'_i) = 1, x'_i \in M'_i \right\} \leq \lambda_k \lambda_{\sigma(k)} K_G^2 \|x\|_{\sigma(\sigma(k))}$$

and the result follows. The inequality

$$(*) \quad \sum_i \|x_i\|_k \leq \lambda_k \lambda_{\sigma(k)} K_G^2 \|x\|_{\sigma(\sigma(k))}$$

will be useful later. $\square$

3. THE MAIN RESULTS

In [9], Wotjinsky posed the next conjecture: “If all the bases of a Fréchet space E with a basis are unconditional, then E is nuclear”, and he proved that the conjecture holds when E is Hilbertisable. Here we provide a positive new approach to the conjecture, showing that it is also true if E is a countably- $\mathcal{L}_1$ space. We need a previous result which generalizes a nice theorem of Lindenstrauss and Pelczynski [3, Theorem 6.1].

Theorem 3.1. *Let E be a countably- $\mathcal{L}_1$ space with an unconditional basis. Then E is isomorphic to an echelon sequence of order 1, $\lambda(a_n^k)$.*

Proof. Indeed, let $(\|\cdot\|_k)_k$ be a f.s.s. of E , let $(e_n)_n$ be an unconditional basis in E , and let us denote by M_n the line spanned by e_n , $n \in \mathbb{N}$. Then $(M_n)_n$ is an unconditional Schauder decomposition of E . From Lemma 2.2 E is isomorphic to $1^1(M_n)$, and it is left to the reader to check that $1^1(M_n)$ is isomorphic to the space $\lambda(a_n^k)$ where $a_n^k = \|e_n\|_k$, for every $n, k \in \mathbb{N}$. $\square$

Now, from Theorem 3.1 and [9, Theorem 5.1] follows our main result.

Corollary 3.2. *Let E be a countably- $\mathcal{L}_1$ space with a basis, such that all bases of E are unconditional. Then E is nuclear.*

4. SOME OTHER APPLICATIONS OF LEMMA 2.2

We study here those countably- $\mathcal{L}_1$ spaces on which a Fréchet lattice structure can be defined.

Theorem 4.1. *Let E be a Fréchet lattice isomorphic to some countably- $\mathcal{L}_1$ space. Then there exists an echelon Köthe space $\Lambda(X, \beta, \mu, g_k)$ order isomorphic to E .*

Proof. Let $(P_k)_k$ be a f.s. of monotonic seminorms defining the topology of E , and let $(Q_k)_k$ be a f.s.s. equivalent to $(P_k)_k$ and so that the local Banach space E_k is an $\mathcal{L}_{1, \lambda_k}$ space for some $\lambda_k \geq 1$ and every $k \in \mathbb{N}$. Without loss of generality we shall assume that $P_k(x) \leq Q_k(x) \leq P_{k+1}(x)$, $k \in \mathbb{N}$, $x \in E$.

Let us define a new system of seminorms on E . Given $k \in \mathbb{N}$, $y \in E$, we put

$$|||y|||_k = \sup \sum_{i=1}^m P_k(y_i)$$

where the supremum is taken over all possible decompositions of $|y|$ on mutually disjoint elements y_i (i.e. $|y| = \sum_{i=1}^m y_i$, $y_i \wedge y_j = 0$, $i \neq j$, $m \in \mathbb{N}$). (We shall prove below that $|||y|||_k < \infty$.) $|||\cdot|||_k$ is a seminorm (the triangle inequality easily follows from the decomposition property [7, p. 53]). Now, let us take any finite set of mutually disjoint elements $(y_i)_{i=1}^m$ and show that they generate an unconditional Schauder decomposition of E . Indeed, note that E , being a countably- $\mathcal{L}_1$ space, is weakly sequentially complete, so it is easy to see that it is countably order complete, hence $x^{\perp\perp}$ is a projection band for all $x \in E$ [7, p. 64, Corollary 2]. Then, let us consider the band projections $R_i: E \rightarrow y_i^{\perp\perp}$, $i = 1, \dots, m$, and define $R_0 := 1_E - \sum_{i=1}^m R_i$; now, it is straightforward to check that $\{y_1^{\perp\perp}, \dots, y_m^{\perp\perp}, R_0(E)\}$ is a (finite-dimensional) unconditional Schauder decomposition of E . Let us find the index $\sigma(k)$ associated to each seminorm $Q_k(\cdot)$ (Lemma 2.1).

$$\begin{aligned} Q_k \left(\sum_{i=0}^m \varepsilon_i R_i x \right) &\leq P_{k+1} \left(\sum_{i=0}^m \varepsilon_i R_i x \right) = P_{k+1} \left(\left| \sum_{i=0}^m \varepsilon_i R_i x \right| \right) \\ &= P_{k+1} \left(\sum_{i=0}^m R_i x \right) = P_{k+1}(x) \leq Q_{k+1}(x). \end{aligned}$$

Thus, $\sigma(k) = k + 1$ and from inequality (*) in Lemma 2.2 we get

$$|||y|||_k \leq \sup \sum_{i=1}^m Q_k(y_i) \leq \sup \lambda_k \lambda_{\sigma(k)} K_G^2 Q_{k+2} \left(\sum_{i=1}^m y_i \right) \leq \lambda_k \lambda_{\sigma(k)} K_G^2 P_{k+3}(y)$$

and so, the system of seminorms $(|||\cdot|||_k)_k$ is equivalent to $(P_k)_k$. To end the proof, note that $|||x + y|||_k \geq |||x|||_k + |||y|||_k$ if $x \wedge y = 0$, for all $k \in \mathbb{N}$, and apply Theorem B. $\square$

As a consequence, we provide a new proof of the next result, which is already known (even if E is a nuclear vector lattice, not necessarily Fréchet [2]).

Corollary 4.2. *Let E be a Fréchet lattice. If E is nuclear then it is discrete.*

Proof. Indeed, E can be regarded as a countably- $\mathcal{L}_1$ space [1, 21.2.5]. Then, from the Theorem above it is order isomorphic to some echelon Köthe space $\Lambda(X, \beta, \mu, g_k)$. From Lemma A, (X, β, μ) is purely atomic, so Λ , and consequently E , are discrete. $\square$

It follows the next characterization of the nuclear Fréchet spaces with a basis.

Corollary 4.3. *A nuclear Fréchet space has a basis iff it is isomorphic to some Fréchet lattice.*

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