A PROOF OF THE FEFFERMAN-STEIN-STRÖMBERG INEQUALITY FOR THE SHARP MAXIMAL FUNCTIONS

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ABSTRACT. We give another proof of a theorem of Strömberg for the Fefferman-Stein sharp maximal functions. Our method is based on a decomposition lemma which is due to the arguments of Carleson, Garnett and Jones for the functions of BMO, and it is valid for a two-weight setting under a condition which is equivalent to the A_∞ condition in the case of equal weights.

1. INTRODUCTION

Let f be a locally integrable function on $\mathbb{R}^n$ and I denotes a cube in $\mathbb{R}^n$ with sides parallel to the axes. Then, if

$$\sup_I \frac{1}{|I|} \int_I |f(x) - f_I| dx < \infty,$$

f is said to be of BMO. Here $|I|$ is the Lebesgue measure of I and $f_I = 1/|I| \int_I f dx$. In [2] Carleson obtained a decomposition theorem for the functions of BMO, and Garnett and Jones [4] have given a new proof of the theorem of Carleson.

In both cases the methods are based on the well-known lemma of Calderón and Zygmund [1]. This lemma is also useful for locally integrable functions which are not necessarily of BMO. By using this decomposition lemma, which will appear as Lemma 1 in this note, we shall give another proof of a theorem of Strömberg [5, Theorem 3.1. (ia), p. 523] for the Fefferman-Stein sharp maximal functions in a two-weight setting.

The sharp maximal function $f^\#(x)$ is defined by

$$f^\#(x) = \sup \frac{1}{|I|} \int_I |f(y) - f_I| dy,$$

where the supremum is taken over all the cubes I containing x . (See Fefferman and Stein [3, p. 153].)

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Let $\varphi(x)$ be a non-negative increasing function on $[0, \infty]$ satisfying that $\varphi(0) = 0$ and

$$(1.1) \quad \varphi(2t) \leq C_0 \varphi(t) \quad \text{for any } t > 0,$$

where the constant C_0 is independent of t . For example, the functions $\varphi(t) = t^p$, $0 < p < \infty$, satisfy (1.1).

The growth condition (1.1) for φ was introduced into sharp maximal function inequalities by Strömberg [5, p. 512].

Now we state our result:

Theorem. Let $w(x)$ and $v(x)$ be non-negative and locally integrable functions on $\mathbb{R}^n$. Suppose there exist positive numbers C_1 , α and β satisfying that $0 < \alpha < 1$, and that for any cube I and a measurable subset E of I

$$(1.2) \quad \int_E w(x) dx \leq C_1 \left(\frac{|E|}{|I|} \right)^\beta \int_{I \setminus E} v(x) dx$$

whenever $|E| \leq \alpha|I|$.

Then there exists a positive constant C_2 , depending only on C_0 , C_1 , α , β and the dimension n , which satisfies that

$$(1.3) \quad \int_{\mathbb{R}^n} \varphi(|f(x)|) w(x) dx \leq C_2 \int_{\mathbb{R}^n} \varphi(f^\#(x)) v(x) dx,$$

for any locally integrable function f , provided that

$$(1.4) \quad \lim_{|Q| \rightarrow \infty} f_Q = 0,$$

where Q are the cubes centered at the origin.

Remark 1. By Lebesgue's theorem on differentiating the integral, (1.2) implies that

$$(1.5) \quad w(x) \leq C_3 v(x) \quad \text{for almost all } x \in \mathbb{R}^n.$$

Remark 2. When $w = v$, the condition (1.2) is equivalent to the A_∞ condition:

$$\int_E w dx \leq C \left(\frac{|E|}{|I|} \right)^\beta \int_I w dx$$

for a cube I and a subset E of I . In this case the theorem may be proved by the methods of Fefferman and Stein [3, pp. 153–155] and Strömberg [5, pp. 524–525].

We give a simple example of a pair (w, v) of functions on $\mathbb{R}$ which satisfies (1.2). We define that $w(x) = 0$ if $0 \leq x < 1$ and $w(x) = 1$ otherwise, and that $v(x) = 0$ if $\frac{1}{3} \leq x < \frac{2}{3}$ and $v(x) = 1$ otherwise. Then we can easily see that (1.2) holds for w and v , but neither of them is an A_∞ weight and we cannot also insert any A_∞ weight between them.

Throughout this note various constants C and C_j depend only on C_0 , C_1 , α , β , and n .

2. PROOF OF THEOREM

We shall begin by stating the lemma. Let f be a locally integrable function on $\mathbb{R}^n$ and let Q be a cube in $\mathbb{R}^n$. We put for x in Q

$$f^{*,Q}(x) = \sup \frac{1}{|I|} \int_I |f(y) - f_I| dy,$$

where the supremum is taken over all the dyadic subcubes I of Q which contain x . Dyadic subcubes I of Q with measure $2^{-jn}|Q|$ are obtained by bisecting each side of Q j times, respectively.

Lemma 1. *Let A be a positive number and f be a locally integrable function. If $|\{x \in Q; |f(x) - f_Q| > Af^{*,Q}(x)\}| \neq 0$ then there exist a measurable function g , a sequence $\{a_\nu^{(j)}\}_{j,\nu}$ of nonzero numbers and families of dyadic subcubes $\mathcal{F}_j = \{I_\nu^j\}_\nu$, $j = 1, 2, \dots$, in Q satisfying that*

- (i) *the cubes I_ν^j in $\mathcal{F}_j$ are pairwise disjoint for each j ,*
- (ii) *$f(x) = f_Q + g(x) + \sum_{j=1}^\infty \sum_\nu a_\nu^{(j)} \chi_{I_\nu^j}(x)$ for almost all x in Q*

where $\chi_{I_\nu^j}(x)$ is the characteristic function of I_ν^j ,

- (iii) *$|g(x)| \leq Af^{*,Q}(x)$ for almost all x in Q ,*
- (iv) *$|a_\nu^{(j)}| \leq (2^n + A) \sup_{I \supseteq I_\nu^j} \frac{1}{|I|} \int_I |f - f_I| dx$,*

where I are dyadic subcubes of Q , and

- (v) *$\bigcup_{I_\mu^{j+1} \in \mathcal{F}_{j+1}} I_\mu^{j+1} \subset \bigcup_{I_\nu^j \in \mathcal{F}_j} I_\nu^j$ for every j ,*

moreover, if $k > j$, then for $I_\nu^j \in \mathcal{F}_j$

$$\sum_{I_\mu^k \subset I_\nu^j, I_\mu^k \in \mathcal{F}_k} |I_\mu^k| \leq A^{-(k-j)} |I_\nu^j|.$$

Remark 3. When f is of BMO, Lemma 1 means Theorem 2.1 of Garnett and Jones [4, pp. 355–356]. And our proof of Lemma 1 will be the same as that of Theorem 2.1 of Garnett and Jones in that case. So we shall sketch the proof of Lemma 1.

Proof of Lemma 1. We shall use the manner of Carleson in [2, pp. 273–276].

Let $\{Q_\nu^j\}_{\nu=1}^{2^j n}$ be the family of all the dyadic subcubes of Q with measure $2^{-jn}|Q|$.

Put $f_1(x) = f(x) - f_Q$. Inductively for every $j \geq 1$, we set

$$b_\nu^{(j)} = (f_j)_{Q_\nu^j} \quad \text{if } |(f_j)_{Q_\nu^j}| > A \sup_{Q_\mu^k \supset Q_\nu^j} \frac{1}{|Q_\mu^k|} \int_{Q_\mu^k} |f(x) - f_{Q_\mu^k}| dx$$

and $b_\nu^{(j)} = 0$ otherwise and we set $f_{j+1}(x) = f_j(x) - \sum_\nu b_\nu^{(j)} \chi_{Q_\nu^j}(x)$.

Then

$$(2.1) \quad |(f_j)_{Q_\mu^{j-1}}| \leq A \sup_{Q_\xi^k \supset Q_\mu^{j-1}} (|f - f_{Q_\xi^k}|)_{Q_\xi^k}$$

and therefore, if $Q_\nu^j \subset Q_\mu^{j-1}$,

$$(2.2) \quad |b_\nu^{(j)}| \leq |(f_j)_{Q_\nu^j} - (f_j)_{Q_\mu^{j-1}}| + |(f_j)_{Q_\mu^{j-1}}| \\ \leq (2^n + A) \sup_{Q_\xi^k \supset Q_\mu^{j-1}} (|f - f_{Q_\xi^k}|)_{Q_\xi^k}.$$

Lebesgue's theorem on differentiating the integral shows that

$$g(x) = \lim_{j \rightarrow \infty} f_j(x)$$

converges for almost all x in Q and (2.1) implies (iii).

Let $\mathcal{F}_1 = \{Q_\nu^k; b_\nu^{(k)} \neq 0 \text{ and maximal}\}$ and inductively we let $\mathcal{F}_j = \{Q_\nu^k; b_\nu^{(k)} \neq 0, Q_\nu^k \notin \mathcal{F}_{j-1} \text{ and maximal}\}$. If $\mathcal{F}_j \neq \emptyset$, we set $a_\nu^{(j)} = b_\nu^{(k)} \neq 0$ and $I_\nu^j = Q_\nu^k$ for Q_ν^k in $\mathcal{F}_j$. Then (2.2) implies (iv).

If $I_\mu^{j+1} \subset I_\nu^j$, then $|a_\mu^{(j+1)}| = |f_{I_\mu^{j+1}} - f_{I_\nu^j}| > A(|f - f_{I_\nu^j}|)_{I_\nu^j}$. Hence

$$(2.3) \quad (|f - f_{I_\nu^j}|)_{I_\nu^j} < \frac{1}{A} (|f - f_{I_\nu^j}|)_{I_\mu^{j+1}}.$$

Summing up (2.3) with respect to $I_\mu^{j+1} \subset I_\nu^j$ we get

$$\sum_{I_\mu^{j+1} \subset I_\nu^j} |I_\mu^{j+1}| \leq \frac{1}{A} |I_\nu^j|.$$

By repeating this argument we have (v).

We shall also need later the following simple lemma:

Lemma 2. Let $\{a^{(j)}\}_{j=1}^k$ be a sequence of numbers. Then

$$\varphi \left(\sum_{j=1}^k |a^{(j)}| \right) \leq \sum_{j=1}^k C_0^{k-j+1} \varphi(|a^{(j)}|),$$

where C_0 is the constant which satisfies (1.1).

The conclusion of Lemma 2 is immediate from the fact that

$$\varphi(|a+b|) \leq C_0\{\varphi(|a|) + \varphi(|b|)\}.$$

Proof of Theorem. Take A to be a positive number larger than $\max\{1/\alpha, C_0^{1/\beta}\}$, and let Q be a cube centered at the origin. Then, by Lemma 1 for the number A and the cube Q we have a function $g(x)$, a sequence $\{a_\nu^{(j)}\}$ of numbers and families $\mathcal{F}_j$ of dyadic subcubes of Q which satisfy (i), (ii), (iii), (iv), and (v) of Lemma 1. (If $|\{x \in Q; |f(x) - f_Q| > A f^\#(x)\}| = 0$, we can put $f(x) = f_Q + g(x)$, where g satisfies (iii).)

If a point x in Q satisfies (ii) and if $|f(x)| \geq 2|f_Q|$, we get

$$|f(x)| \leq 2|g(x) + \sum_{j,\nu} a_\nu^{(j)} \chi_{I'_\nu}(x)|.$$

Therefore, we have from (1.1)

$$\begin{aligned} (2.4) \quad & \int_{Q \cap \{|f(x)| \geq 2|f_Q|\}} \varphi(|f(x)|) w(x) dx \\ & \leq C_0 \int_Q \varphi(|g(x) + \sum_{j,\nu} a_\nu^{(j)} \chi_{I'_\nu}(x)|) w(x) dx \\ & \leq C_0^2 \left\{ \int_Q \varphi(|g(x)|) w(x) dx + \int_Q \varphi \left(\sum_{j,\nu} |a_\nu^{(j)}| \chi_{I'_\nu}(x) \right) w(x) dx \right\}. \end{aligned}$$

Using (iii), (1.1) and (1.5) we obtain

$$(2.5) \quad \int_Q \varphi(|g(x)|) w(x) dx \leq C_3 C_0^{\log A} \int_Q \varphi(f^{\#,Q}(x)) v(x) dx.$$

Next we shall show the following estimate:

$$(2.6) \quad \int_Q \varphi \left(\sum_{j=1}^{\infty} \sum_{\nu} |a_\nu^{(j)}| \chi_{I'_\nu}(x) \right) w(x) dx \leq C_4 \int_Q \varphi(f^{\#,Q}(x)) v(x) dx.$$

From (v) we can divide the integration domain of the left-hand side of (2.6) as follows:

$$\begin{aligned} (2.7) \quad & \int_Q \varphi \left(\sum_{j=1}^{\infty} \sum_{\nu} |a_\nu^{(j)}| \chi_{I'_\nu}(x) \right) w dx \\ & = \int_{\cup_{\mu} I_{\mu}^1 \setminus \cup_{\xi} I_{\xi}^2} \varphi \left(\sum_{\nu} |a_\nu^{(1)}| \chi_{I'_\nu}(x) \right) w dx \\ & \quad + \sum_{k=2}^{\infty} \int_{\cup_{\mu} I_{\mu}^k \setminus \cup_{\xi} I_{\xi}^{k+1}} \varphi \left(\sum_{j=1}^k \sum_{\nu} |a_\nu^{(j)}| \chi_{I'_\nu}(x) \right) w dx, \end{aligned}$$

where $I_{\mu}^k \in \mathcal{F}_k$ and $I_{\xi}^{k+1} \in \mathcal{F}_{k+1}$. For every $k \geq 2$ we observe

$$\begin{aligned} (2.8) \quad & \int_{\cup_{\mu} I_{\mu}^k \setminus \cup_{\xi} I_{\xi}^{k+1}} \varphi \left(\sum_{j=1}^k \sum_{\nu} |a_\nu^{(j)}| \chi_{I'_\nu}(x) \right) w dx \\ & \leq \sum_{j=1}^k C_0^{k-j+1} \int_{\cup_{\mu} I_{\mu}^k \setminus \cup_{\xi} I_{\xi}^{k+1}} \varphi \left(\sum_{\nu} |a_\nu^{(j)}| \chi_{I'_\nu}(x) \right) w dx \end{aligned}$$

(by Lemma 2)

$$\begin{aligned} &\leq C_0 \int_{\bigcup_{\mu} I_{\mu}^k \setminus \bigcup_{\xi} I_{\xi}^{k+1}} \varphi \left(\sum_{\nu} |a_{\nu}^{(k)}| \chi_{I_{\nu}^k} \right) w \, dx \\ &\quad + \sum_{j=1}^{k-1} C_0^{k-j+1} \int_{\bigcup_{\mu} I_{\mu}^k} \varphi \left(\sum_{\nu} |a_{\nu}^{(j)}| \chi_{I_{\nu}^j} \right) w \, dx. \end{aligned}$$

From (v) and the hypothesis (1.2) it follows that

$$\begin{aligned} &\int_{\bigcup_{\mu} I_{\mu}^k} \varphi \left(\sum_{\nu} |a_{\nu}^{(j)}| \chi_{I_{\nu}^j} \right) w \, dx \\ &\leq C_1 A^{-\beta(k-j)} \int_{\bigcup_{\nu} I_{\nu}^j \setminus \bigcup_{\mu} I_{\mu}^k} \varphi \left(\sum_{\nu} |a_{\nu}^{(j)}| \chi_{I_{\nu}^j} \right) v \, dx. \end{aligned}$$

And from (1.5) we have for $k \geq 2$ and also for $k = 1$

$$\begin{aligned} &\int_{\bigcup_{\mu} I_{\mu}^k \setminus \bigcup_{\xi} I_{\xi}^{k+1}} \varphi \left(\sum_{\nu} |a_{\nu}^{(k)}| \chi_{I_{\nu}^k} \right) w \, dx \\ &\leq C_3 \int_{\bigcup_{\mu} I_{\mu}^k \setminus \bigcup_{\xi} I_{\xi}^{k+1}} \varphi \left(\sum_{\nu} |a_{\nu}^{(k)}| \chi_{I_{\nu}^k} \right) v \, dx. \end{aligned}$$

Therefore, we see that the right-hand side of (2.8) is majorized by

$$\begin{aligned} &C_5 \left\{ \int_{\bigcup_{\mu} I_{\mu}^k \setminus \bigcup_{\xi} I_{\xi}^{k+1}} \varphi \left(\sum_{\nu} |a_{\nu}^{(k)}| \chi_{I_{\nu}^k} \right) v \, dx \right. \\ &\quad \left. + C_1 \sum_{j=1}^{k-1} (C_0 A^{-\beta})^{k-j} \int_{\bigcup_{\nu} I_{\nu}^j \setminus \bigcup_{\mu} I_{\mu}^k} \varphi \left(\sum_{\nu} |a_{\nu}^{(j)}| \chi_{I_{\nu}^j} \right) v \, dx \right\}. \end{aligned}$$

Summing up the above expression with respect to k and using (iv) we see that the left-hand side of (2.7) is majorized by

$$\begin{aligned} (2.9) \quad &C_5 \left\{ \sum_{k=1}^{\infty} \int_{\bigcup_{\mu} I_{\mu}^k \setminus \bigcup_{\xi} I_{\xi}^{k+1}} \varphi \left(\sum_{\nu} |a_{\nu}^{(k)}| \chi_{I_{\nu}^k} \right) v \, dx \right. \\ &\quad \left. + C_1 \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} (C_0 A^{-\beta})^{k-j} \int_{\bigcup_{\nu} I_{\nu}^j \setminus \bigcup_{\mu} I_{\mu}^k} \varphi \left(\sum_{\nu} |a_{\nu}^{(j)}| \chi_{I_{\nu}^j} \right) v \, dx \right\} \\ &\leq C_6 \left\{ \int_Q \varphi(f^{\#, Q}(x)) v \, dx \right. \\ &\quad \left. + C_1 \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} (C_0 A^{-\beta})^{k-j} \int_{\bigcup_{\nu} I_{\nu}^j \setminus \bigcup_{\mu} I_{\mu}^k} \varphi(f^{\#, Q}(x)) v \, dx \right\}. \end{aligned}$$

When we interchange the order of summation, the second term of the last expression becomes

$$\begin{aligned} & C_1 \sum_{j=1}^{\infty} \sum_{k=j+1}^{\infty} (C_0 A^{-\beta})^{k-j} \int_{\cup_{\nu} I_{\nu}^j \setminus \cup_{\mu} I_{\mu}^k} \varphi(f^{\#,Q}(x)) v \, dx \\ &= C_1 \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} (C_0 A^{-\beta})^l \int_{\cup_{\nu} I_{\nu}^j \setminus \cup_{\mu} I_{\mu}^{l+j}} \varphi(f^{\#,Q}(x)) v \, dx \\ &= C_1 \sum_{l=1}^{\infty} (C_0 A^{-\beta})^l \sum_{j=1}^{\infty} \int_{\cup_{\nu} I_{\nu}^j \setminus \cup_{\mu} I_{\mu}^{l+j}} \varphi(f^{\#,Q}(x)) v \, dx. \end{aligned}$$

Since the integration domains $\{\cup_{\nu} I_{\nu}^j \setminus \cup_{\mu} I_{\mu}^{l+j}\}_j$ are overlapping at most l times for each l , the above expression is bounded by

$$C_1 \sum_{l=1}^{\infty} l (C_0 A^{-\beta})^l \int_Q \varphi(f^{\#,Q}(x)) v \, dx.$$

Because $C_0 A^{-\beta} < 1$ we conclude that (2.9) is majorized by $C \int_Q \varphi(f^{\#,Q}(x)) v \, dx$.

Thus we get (2.6). And (2.4), (2.5), and (2.6) imply that

$$\int_{Q \cap \{|f(x)| \geq 2|f_Q|\}} \varphi(|f(x)|) w \, dx \leq C_7 \int_{\mathbb{R}^n} \varphi(f^{\#}(x)) v \, dx.$$

When $|Q| \rightarrow \infty$, we obtain the conclusion (1.3) of the theorem by the hypothesis (1.4). This completes the proof of the theorem.

I would like to add that if we replace f_Q by the median value $m_f(Q)$, then the theorem holds for the sharp maximal functions $M_{0,1/A}^{\#} f(x)$, where A is sufficiently large.

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