PRIMITIVE ELEMENTS IN FREE GROUPS

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ABSTRACT. Let F_n denote the free group of rank n generated by $x_1, x_2, \dots, x_n$. We say that $y \in F_n$ is a *primitive* element of F_n if it is contained in a set of free generators of F_n . In this note we construct, for each integer $n \geq 4$, an $(n-1)$ -generator group H that has an n -generator, 2-relator presentation $H = \langle x_1, \dots, x_n | r_1, r_2 \rangle$ such that the normal closure of $\{r_1, r_2\}$ in F_n does not contain a primitive element of F_n .

1. INTRODUCTION

Let F_n denote the free group of rank n , freely generated by $x_1, x_2, \dots, x_n$. We say that $x \in F_n$ is a *primitive* element of F_n if it is contained in a set of free generators of F_n .

According to [2, p. 92], F. Waldhausen asked "If $G = \langle x_1, x_2, \dots, x_n | r_1, \dots, r_m \rangle$ can be generated by fewer than n elements, then does the normal closure of $\{r_1, \dots, r_m\}$ in F_n contain a primitive element of F_n ?" A negative answer has been obtained by G. A. Noskov [5]. His ingenious argument is partly geometric and uses the "Hairy Ball" theorem at a crucial point.

The purpose of this note is to prove the following.

Theorem. *For each integer $n \geq 4$, there exists an $(n-1)$ -generator group H that has an n -generator, 2-relator presentation $H = \langle x_1, \dots, x_n | r_1, r_2 \rangle$ such that the normal closure of $\{r_1, r_2\}$ in F_n does not contain a primitive element of F_n .*

We remark that any presentation that provides a negative answer to Waldhausen's question has at least two relators. For suppose that $G = \langle x_1, \dots, x_n | r \rangle$ can be generated by fewer than n elements. It follows easily from [3, Corollary 5.14.2] that G is free and we can now use a result of J. H. C. Whitehead (see [2, Chapter 2], Proposition 5.10) to deduce that r is a primitive element of F_n .

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2. THE MODULES M AND B

Let $G = \langle x, y | x^2 = y^3 \rangle$; the trefoil group. M. J. Dunwoody [1] has shown that $M = \mathbb{Z}G \oplus \mathbb{Z}G$, the free right G -module of rank 2, can be decomposed as $M = C \oplus B$ where $C \cong \mathbb{Z}G$ and B cannot be generated (as a G -module) by a single element. With this notation we have

Lemma 1. *M cannot be generated (as a G -module) by two elements, one of which lies in B .*

Proof. It is convenient to view M as an external direct sum of $\mathbb{Z}G$ and B . Thus $M = \{(z, b) | z \in \mathbb{Z}G, b \in B\}$.

Suppose, for a contradiction, that M is generated by (z, b) and $(0, b')$. Note that b and b' together generate B . Now there exist $\alpha, \beta \in \mathbb{Z}G$ such that $(z, b)\alpha + (0, b')\beta = (1, 0)$. So $z\alpha = 1$ and $b\alpha = -b'\beta$. Since $z\alpha = 1$, a result of Kaplansky (see [4]) implies that $\alpha z = 1$. Therefore $b = -b'\beta z$ and B is generated by b' . However, by the preceding remarks, B cannot be generated by a single element and we have obtained the desired contradiction.

Since $M = \mathbb{Z}G \oplus \mathbb{Z}G$, M can be generated by two elements, m_1 and m_2 , say. Furthermore, B is an image of M so B can be generated by two elements, b_1 and b_2 . Also, $C \cong \mathbb{Z}G$, so C can be generated by a single element c . Note that M is generated by c, b_1 and b_2 . We fix this notation once and for all.

3. THE GROUPS W AND $\overline{W}$

From now on, n denotes a fixed integer with $n \geq 4$. Let F denote the free group of rank $n - 2$, generated by $y_1, y_2, \dots, y_{n-2}$, and let $\theta: F \rightarrow G$ be any epimorphism.

Define an action of F on M by $m^f = m^{f\theta}$, for all $m \in M$ and $f \in F$. Note that $K \subseteq M$ is an F -submodule of M if and only if it is a G -submodule of M , so we may write unambiguously of submodules of M . Combining this remark with Lemma 1 yields

Lemma 2. *M cannot be generated as an F -module by two elements, one of which lies in B .*

Since M is an F -module we can form $W = M \rtimes F$, the semidirect product of M and F . We shall identify M and F with their natural copies in W so that $W = MF$ and $M \cap F$ is trivial.

Recall that the elements m_1 and m_2 generate M , so $W = \langle y_1, \dots, y_{n-2}, m_1, m_2 \rangle$ and W is n -generator.

Lemma 3. *W cannot be generated by n elements, one of which lies in B .*

Proof. Let A be any group. The following transformations of the set of s -tuples $(a_1, a_2, \dots, a_s)$, where $a_1, a_2, \dots, a_s \in A$, are called elementary Nielsen transformations: (i) permuting the a_i 's (ii) inverting a_i for some i (iii) leaving a_i

fixed for $i \neq j$ and replacing a_j with $a_k^{\pm 1} a_j$ or $a_j a_k^{\pm 1}$ for some $k \neq j$. Observe that if σ is an elementary Nielsen transformation and $(a_1, a_2, \dots, a_s)\sigma = (a'_1, a'_2, \dots, a'_s)$ then $\langle a_1, a_2, \dots, a_s \rangle = \langle a'_1, a'_2, \dots, a'_s \rangle$.

Now suppose that the lemma is false so that $W = \langle h_1 f_1, h_2 f_2, \dots, h_{n-1} f_{n-1}, b \rangle$ where $h_1, \dots, h_{n-1} \in M$, $f_1, \dots, f_{n-1} \in F$ and $b \in B$. Clearly $\langle f_1, f_2, \dots, f_{n-1} \rangle = F$. It is well known (e.g. [3, Chapter 3]) that there exists a finite sequence of elementary Nielsen transformations $\sigma_1, \sigma_2, \dots, \sigma_l$ such that $(f_1, f_2, \dots, f_{n-1})\sigma_1 \sigma_2 \dots \sigma_l = (y_1, y_2, \dots, y_{n-2}, 1)$. Apply the same transformation to $(h_1 f_1, h_2 f_2, \dots, h_{n-1} f_{n-1})$, we obtain

$$(h_1 f_1, h_2 f_2, \dots, h_{n-1} f_{n-1})\sigma_1 \sigma_2 \dots \sigma_l = (k_1 y_1, k_2 y_2, \dots, k_{n-1} y_{n-2}, k_{n-1})$$

for some $k_1, k_2, \dots, k_{n-1} \in M$. It follows that $\langle h_1 f_1, h_2 f_2, \dots, h_{n-1} f_{n-1} \rangle = \langle k_1 y_1, k_2 y_2, \dots, k_{n-2} y_{n-2}, k_{n-1} \rangle$ so that

$$W = \langle k_1 y_1, k_2 y_2, \dots, k_{n-2} y_{n-2}, k_{n-1}, b \rangle.$$

Let K denote the submodule of M generated by $k_1, \dots, k_{n-2}$ and let $S = \langle k_1 y_1, \dots, k_{n-2} y_{n-2} \rangle$. Now $S/(K \cap S) \cong KS/K = KF/K \cong F/(F \cap K) \cong F$ so S has a free group of rank $n-2$ as a homomorphic image. However, S can be generated by $n-2$ elements and it follows that S is free of rank $n-2$.

Now let N denote the submodule of M generated by k_{n-1} and b . We have $W = NS$ so that $M = M \cap NS = N(M \cap S)$. Since S is free of rank $n-2$ it has no nontrivial abelian normal subgroups, so $M \cap S$ is trivial and we deduce that $N = M$. Thus M is generated by two elements k_{n-1} and b , one of which lies in B . This contradicts Lemma 2 and completes the proof.

Now B is a submodule of M so $B \triangleleft W$. Define $\overline{W} = W/B$ and write $\overline{w}$ for the image of $w \in W$ under the natural homomorphism $\pi: W \rightarrow W/B$. Since M is generated by c, b_1 and b_2 , we have $W = \langle y_1, \dots, y_{n-2}, c, b_1, b_2 \rangle$. Therefore $\overline{W} = \langle \overline{y}_1, \dots, \overline{y}_{n-2}, \overline{c} \rangle$ and $\overline{W}$ is $(n-1)$ -generator.

In light of Lemma 4 below, it is interesting to observe that $\overline{W}$ is isomorphic to the semi-direct product of $\mathbf{Z}G$ and F , where F acts on $\mathbf{Z}G$ via θ .

Our final lemma also established some notation that will be used in the proof of the theorem.

Lemma 4. *Let*

$\eta: F_n \rightarrow W$ *be the epimorphism defined by*

$\eta: x_i \rightarrow y_i$ *for* $i = 1, 2, \dots, n-2$,

$\eta: x_{n-1} \rightarrow m_1$,

$\eta: x_n \rightarrow m_2$.

Furthermore, let π *denote the natural homomorphism* $\pi: W \rightarrow W/B = \overline{W}$ *and define an epimorphism* $\phi: F_n \rightarrow \overline{W}$ *by* $\phi = \eta\pi$ *so that*

$\phi: x_i \rightarrow \overline{y}_i$ $i = 1, 2, \dots, n-2$,

$\phi: x_{n-1} \rightarrow \overline{m}_1$,

$\phi: x_n \rightarrow \overline{m}_2$.

Then $\ker \phi$ does not contain a primitive element of F_n .

Proof. Suppose $p \in \ker \phi$ is a primitive element of F_n . Then there exist $p_1, p_2, \dots, p_{n-1} \in F_n$ such that $\langle p_1, p_2, \dots, p_{n-1}, p \rangle = F_n$. Now $W = \langle p_1\eta, p_2\eta, \dots, p_{n-1}\eta, p\eta \rangle$ where $p\eta\pi = p\phi = 1$ so that $p\eta \in \ker \pi = B$. Thus W can be generated by n elements, one of which lies in B . This contradicts Lemma 3 and the result follows.

We point out that the significance of Lemma 4 is that the $(n-1)$ -generator group $\overline{W}$ can be written as $\overline{W} = F_n / \ker \phi$, where $\ker \phi$ does not contain a primitive element of F_n .

4. PROOF OF THE THEOREM

Recall that $\overline{W} = \langle \overline{y}_1, \overline{y}_2, \dots, \overline{y}_{n-2}, \overline{c} \rangle$ and let z be a preimage of $\overline{c}$ under ϕ .

Consider $L = \langle x_1, x_2, \dots, x_{n-2}, z \rangle$, a subgroup of F_n . Let ϕ_L denote the restriction of ϕ to L and note that $\phi_L: L \rightarrow \overline{W}$ is onto since $x_i\phi_L = \overline{y}_i$ for $i = 1, 2, \dots, n-2$ and $z\phi_L = \overline{c}$. Therefore, there exist $z_1, z_2 \in L$ such that $z_1\phi_L = \overline{m}_1$ and $z_2\phi_L = \overline{m}_2$.

We shall show that $H = \langle x_1, \dots, x_n | x_{n-1}z_1^{-1}, x_nz_2^{-1} \rangle$ has the properties referred to in the statement of the theorem.

Let R denote the normal closure of $\{x_{n-1}z_1^{-1}, x_nz_2^{-1}\}$ in F_n so that $H \cong F_n/R$. Firstly, $x_{n-1}R = z_1R$ and $x_nR = z_2R$ where $z_1, z_2 \in \langle x_1, x_2, \dots, x_{n-2}, z \rangle$ so that F_n/R is generated by $x_1R, x_2R, \dots, x_{n-2}R$ and zR . Therefore H is $(n-1)$ -generator. Moreover, $z_1\phi = \overline{m}_1 = x_{n-1}\phi$ and $z_2\phi = \overline{m}_2 = x_n\phi$ so $x_{n-1}z_1^{-1}, x_nz_2^{-1} \in \ker \phi$. Therefore $R \leq \ker \phi$ and it follows immediately from Lemma 4 that R does not contain any primitive elements of F_n . The theorem is proved.

REFERENCES

1. M. J. Dunwoody, *Relation modules*, Bull. London Math. Soc. **4** (1972), 151–155.
2. R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Springer-Verlag, Berlin-Heidelberg-New York, 1977.
3. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory*, Interscience, New York, 1966.
4. M. S. Montgomery, *Left and right inverses in group algebras*, Bull. Amer. Math. Soc. **75** (1969), 539–540.
5. G. A. Noskov, *Primitive elements in a free group*, Mat. Zametki **30**(4) (1981), 497–500.

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