A NOTE ON THE VARIETY OF PLANE CURVES WITH NODES AND CUSPS

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ABSTRACT. Let $\mathbf{P}^N$ be the projective space of all degree d plane curves and $V(d, \delta, \kappa)$ the subvariety of $\mathbf{P}^N$ consisting of all reduced and irreducible curves with δ nodes and κ cusps. Then $V(d, \delta, \kappa)$ is irreducible if $\kappa \leq \frac{1}{2}(d+1) - g$.

Let $\mathbf{P}^N$ be the projective space parametrizing all plane curves of degree d and $V(d, \delta, \kappa)$ the subvariety of $\mathbf{P}^N$ corresponding to all reduced and irreducible plane curves of degree d with δ nodes and κ ordinary cusps as their only singular points. Diaz and Harris [DH] showed that $V(d, \delta, \kappa)$ is a smooth variety of dimension $N - \delta - 2\kappa$ if $\kappa \leq 2d - 1$. In [K] we showed that $V(d, \delta, \kappa)$ is irreducible if $\kappa \leq 3$ except possibly when $\kappa = 3$ and $d = 5$ or 6.

In this paper we prove

Theorem. *The variety $V(d, \delta, \kappa)$ is irreducible if $\kappa \leq \frac{1}{2}(d+1) - g$.*

This result goes along with the general suspicion that $V(d, \delta, \kappa)$ should be irreducible when κ is pretty small with respect to d while κ approaches infinity as d does. We remark here that there exist two components of $V(6, 0, 6)$ [Z].

We mean here by a node a point where a curve has two transverse smooth branches, and by a cusp a point where a curve has a singular branch, if it is locally given by $y^2 = x^3$ we call it an ordinary cusp. The (geometric) genus of a curve is the genus of its normalization.

From now on we assume that d is at least 5 since the irreducibility of all possible $V(d, \delta, \kappa)$ is known when $d < 5$. We also work on the assumption that $\kappa \neq 0$; otherwise it is the Severi variety $V(d, \delta)$, the variety of reduced and irreducible plane curves of degree d with δ nodes as their only singularities, which is irreducible [H] [Z].

Note that a plane curve of degree d and geometric genus g can be realized as a degree d map $\varphi: C \rightarrow \mathbf{P}^2$, or equivalently, as a 3-dimensional subspace (a

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g_d^2 , for short) of $H^0(\mathcal{L})$ for a line bundle $\mathcal{L}$ of degree d on a smooth genus g curve C . From this point of view we consider $G_{d,\kappa}^2(C)$ the set of all g_d^2 's on a smooth curve C the associated maps $\varphi: C \rightarrow \mathbf{P}^2$ of which give rise to plane curves of degree d with κ cusps. Let $\mathcal{M}_g$ be the moduli space of genus g curves and G parametrize pairs (C, V) , where $C \in \mathcal{M}_g$ and $V \in G_{d,\kappa}^2(C)$. Then the theorem follows once we show that G is irreducible and its general members give rise to plane curves in $V(d, \delta, \kappa)$. Since $\mathcal{M}_g$ is irreducible, it comes from the following lemma and its proof.

Lemma. *Let C be a smooth curve of genus g . Then $G_{d,\kappa}^2(C)$ is irreducible of dimension $3d - 2g - 6 - \kappa$ if $d \geq 2g - 1 + 2\kappa$. If $V \in G_{d,\kappa}^2(C)$ is general and $\varphi_V: C \rightarrow \mathbf{P}^2$ is the associated map, then $\varphi_V(C)$, as a plane curve, has κ ordinary cusps and nodes for the remaining singular points.*

Proof. Let $G_{d,\kappa}^2(C, \mathcal{L})$ be all the g_d^2 's associated to a fixed line bundle $\mathcal{L}$ of degree d , $d \geq 2g - 1 + 2\kappa$, on C . We first show that it is irreducible.

Let $\varphi_{\mathcal{L}} = \varphi: C \rightarrow \mathbf{P}^{d-g} = \mathbf{P}(H^0(\mathcal{L}))$ be an embedding associated to $\mathcal{L}$. We may regard a linear series V of dimension 2 (i.e., $\varphi_V: C \rightarrow \mathbf{P}^2$) associated to $\mathcal{L}$ as giving a rational map from C to $\mathbf{P}(V)$, which is the composition of φ and projection from the center $\mathbf{P}(V)^\perp \simeq \mathbf{P}^{d-g-3}$. Then by identifying C with $\varphi(C)$, $G_{d,\kappa}^2(C, \mathcal{L})$ can be considered as the set of all $(d - g - 3)$ -planes meeting κ tangent lines of C transversely away from C .

Let C_κ be the κ th symmetric product of C and Δ the diagonal of C_κ , i.e., $\Delta = \{(p_1, \dots, p_\kappa) \in C_\kappa \mid p_i = p_j \text{ for some distinct } i \text{ and } j\}$. Then $G_{d,\kappa}^2(C, \mathcal{L})$ becomes a fiber space with a fiber $F(p_1, \dots, p_\kappa)$, over a point $(p_1, \dots, p_\kappa) \in C_\kappa \setminus \Delta$, the set of all $(d - g - 3)$ -planes meeting the tangent lines t_i of C at p_i away from C .

Let $W_3(x_i)$ be all the $(d - g - 3)$ -planes containing a point $x_i \in t_i$, which is codimension 3 in $\text{Gr}(d - g - 3, d - g)$, the Grassmannian space of all $(d - g - 3)$ -planes in $\mathbf{P}^{d-g}$. Then

$$F(p_1, \dots, p_\kappa) = \bigcup_{\{(x_1, \dots, x_\kappa) \mid x_i \in t_i \setminus \{p_i\}\}} \bigcap_{i=1}^{\kappa} W_3(x_i).$$

We now claim that $\bigcap_{i=1}^{\kappa} W_3(x_i)$ is irreducible of dimension $3(d - g - 2 - \kappa)$ and its closure is isomorphic to $\text{Gr}(d - g - 3 - \kappa, d - g - \kappa)$ regardless of the choice of p_i and x_i . Then $F(p_1, \dots, p_\kappa)$ has a natural fiber space structure over κ products of affine lines with an irreducible fiber $\bigcap_{i=1}^{\kappa} W_3(x_i)$. Therefore $F(p_1, \dots, p_\kappa)$ is irreducible, so is $G_{d,\kappa}^2(C, \mathcal{L})$.

It now remains to prove the above claim. Note that the closure of $\bigcap_{i=1}^{\kappa} W_3(x_i)$ is the variety of all $(d - g - 3)$ -planes containing the plane that is spanned by $\{x_1, \dots, x_\kappa\}$. Observe that any κ tangent or secant lines are independent, i.e., they span $(2\kappa - 1)$ -plane; if not some κ tangent or secant lines would span

at most $(2\kappa - 2)$ -plane meeting C at 2κ points counting multiplicities, so the dimension of the hyperplanes containing these 2κ points would be $d - g - 2\kappa - 1$. On the other hand $h^0(\mathcal{L} \otimes \mathcal{O}_C(-2\kappa[pt.])) = d - 2\kappa - g + 1$, which implies that the dimension of the hyperplanes containing $2\kappa + 2$ points is $d - 2\kappa - g$. Then this observation tells that $\{x_1, \dots, x_\kappa\}$ spans $(\kappa - 1)$ -plane S_κ ; otherwise $\{x_1, \dots, x_\kappa, p_1, \dots, p_\kappa\}$ would span at most $(2\kappa - 2)$ -plane meeting C at 2κ points counting multiplicities. Then the set of all $(d - g - 3)$ -planes in $\mathbf{P}^{d-g}$ can be identified with $\text{Gr}(d - g - \kappa - 3, \mathbf{P}^{d-g-\kappa})$ which is irreducible of dimension $3(d - g - 2 - \kappa)$: consider the projection of $\mathbf{P}^{d-g}$ from the center S_κ (cf. [K] Theorem (2.1)).

The irreducibility of $G_{d,\kappa}^2(C)$ follows from the facts, first, that $\text{Pic}^d(C) \simeq \text{Jac}(C)$ is irreducible, second, that each fiber $G_{d,\kappa}^2(C, \mathcal{L})$ over $\mathcal{L} \in \text{Pic}^d(C)$ is irreducible.

For the last part of the lemma, note that, if $F_i = F_i^0 \subset F_i^1 \subset \dots \subset F_i^{d-g}$ is the flag of the osculating planes of C at p_i and $\pi_\Lambda(C)$ has an ordinary cusp at $\pi_\Lambda(p_i)$, then Λ satisfies $\dim(\Lambda \cap F_i^{3+j}) = j$, $j \geq 1$, and $\dim(\Lambda \cap F_i^j) = 0$, $j \leq 3$ (cf. [EH] and [GH]). Such Λ 's then clearly form a general locus of $\bigcap_{i=1}^\kappa W_3(x_i)$. For the nodes part, look at the projection $\pi_\kappa: \mathbf{P}^{d-g} \rightarrow \mathbf{P}^{d-g-\kappa}$ from S_κ and the chodal variety $\mathcal{E}_\kappa(C)$ of $\pi_\kappa(C)$. Bertini then tells that a general $(d - g - \kappa - 3)$ -plane in $\mathbf{P}^{d-g-\kappa}$ must meet $\mathcal{E}_\kappa(C)$ at the right number of smooth points of $\mathcal{E}_\kappa(C)$ (and the locus of smooth points of $\mathcal{E}_\kappa(C)$ is contained in the union of the secant lines of $\pi_\kappa(C)$). Q.E.D.

Remark. In fact, the last part of the lemma follows also from the theorem of Diaz and Harris which is mentioned in the beginning of this paper (see [DH] for details).

Harris pointed out that there was another obvious case where $V(d, \delta, \kappa)$ is irreducible. Consider $V(d, \delta, \kappa)$. Since fixing each cusp and its tangential direction gives 5 linear conditions on the coefficients of plane curves of degree d , $V(d, \delta, \kappa)$ can be considered as a fiber space over $\mathbf{P}^{2\kappa} \times \mathbf{P}^\kappa$ with a linear space fiber for big d .

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