A FUNCTIONAL EQUATION FOR THE JOUKOWSKI TRANSFORMATION

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Dedicated to Ann H. on her 21st birthday

ABSTRACT. The purpose of this paper is to solve a functional equation which arises from the Joukowski transformation. The functional equation has a simple geometric interpretation which is given.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

In the paper [1, Proposition 2.1], it was observed that the transformation

$$j(z) = \frac{1}{2}(z + z^{-1}) \quad (z \neq 0)$$

satisfies the functional equation

$$(1) \quad 2j(|z|) = |j(z) + 1| + |j(z) - 1| \quad \text{for } z \in D,$$

where here and throughout $D := \mathbb{C} \setminus \{0\}$. The transformation j is said to be the Joukowski transformation and has applications in complex analysis and in aerodynamics. By (1) we get that the set $j(\{|z| = r\})$ is the ellipse with the great semiaxis equal to $j(r)$ and foci at -1 and $+1$, if $r \neq 1$ or the line segment $[-1, 1]$, if $r = 1$. H. Haruki in his papers [2], [3] researched other equations arising from the Joukowski transformation.

The main result of this paper is the following theorem.

Theorem 1. *Suppose that a function f is holomorphic in D and satisfies the functional equation*

$$(2) \quad 2f(|z|) = |f(z) + 1| + |f(z) - 1| \quad \text{for } z \in D.$$

Then either $f = \text{const} \geq 1$ or there exist $a > 0$, $p \in \mathbb{N}$ such that

$$f(z) = j((az)^p) \quad \text{for } z \in D.$$

Let T_n denote n -th Chebyshev polynomial for $n \in \mathbb{N}$. It is easy to prove that

$$(3) \quad T_n(j(z)) = j(z^n) \quad \text{for } z \in D, n \in \mathbb{N}.$$

As a generalization of Theorem 1 we have the following:

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Theorem 2. Fix $n \in \mathbb{N}$ and consider the equation

$$(4) \quad 2T_n \circ f(|z|) = |T_n \circ f(z) + 1| + |T_n \circ f(z) - 1|, \quad z \in D.$$

If f is a holomorphic function in D which satisfies (4) then either $f = \text{const} = C$, where C has the form $C = \cos(2\pi k/n)$ for some $0 \leq k \leq [n/2]$ or $C = j(\varepsilon C_1)$, where $C_1 \geq 1$, $\varepsilon \in \mathbb{C}$ and $\varepsilon^n = 1$, or there exist constants $a > 0$, $p \in \mathbb{N}$, $\varepsilon \in \mathbb{C}$, $\varepsilon^n = 1$ such that

$$f(z) = j(\varepsilon(az)^p) \text{ for } z \in D.$$

2. PROOF OF THEOREM

If $f = \text{const} = C$ then, by (2), must be $C \geq 1$. Assume that $f \neq \text{const}$. Then $f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$, where the Laurent series is absolutely convergent for all $z \in D$ and $a_n \neq 0$ for $n \neq 0$. If $r > 0$ then, by (2), $f(r) \geq 1$ and $a_n \in \mathbb{R}$ for every $n \in \mathbb{Z}$. Hence $f(-r) \in \mathbb{R}$ for $r > 0$. First we shall prove

Lemma 1. The function f is even or is odd.

Proof. There exists $r_0 > 0$ such that $f(-r_0) \notin [-1, 1]$, because, if $f(-r) \in [-1, 1]$ for all $r > 0$ then we must have $f(r) = 1$, $r > 0$. By the identity property we have $f = \text{const} = 1$ and this is contrary to the assumption that $f \neq \text{const}$. Since f is continuous there exist $0 < \alpha < r_0 < \beta$ such that $f(-r) \notin [-1, 1]$ for $\alpha < r < \beta$ and $|f(-r)| = f(r)$. This implies that $f(-r) = f(r)$ for $\alpha < r < \beta$ or $f(-r) = -f(r)$ for $\alpha < r < \beta$. By the identity property, $f(-z) = f(z)$ for $z \in D$ or $f(-z) = -f(z)$ for $z \in D$.

Next we prove an important fact.

Lemma 2. There exists $b > 0$ such that $f(b) = 1$.

Proof. The proof is by contradiction. Assume that $f(r) > 1$ for all $r > 0$. Let $\alpha := \inf\{f(r) : r > 0\}$, $\beta := \inf\{f(r) : 0 < r \leq 1\}$, $\gamma := \inf\{f(r) : r > 1\}$. Consider two cases: (i) $\alpha > 1$; (ii) $\alpha = 1$. Let $f(z) = f_1(z) + f_2(z)$ where f_1 and f_2 are the singular part and the regular part of Laurent's expansion, respectively. The functions $f_2(z)$ and $f_1(1/z)$ are entire. Suppose that $\alpha > 1$. By (2) we have $|f(z)| \geq f(|z|) - 1 \geq \alpha - 1 > 0$. If $f_1(1/z)$ and $f_2(z)$ are polynomials then the function $g(z) = z^k f(z)$, $k = \deg f_1(1/z)$, is a polynomial and $g(0) \neq 0$, $|g(z)| \geq |z|^k(\alpha - 1) > 0$ for $z \neq 0$. This means that $g = \text{const} = C$ and $f(z) = Cz^{-k}$, where $C = f(1)$. For $r = (f(1))^{1/k}$ we have $f(r) = 1$. If $f_1(1/z)$ or $f_2(z)$ is a transcendental entire function then $z = 0$ or $z = \infty$ is an essential singularity of f . By the Great Picard's Theorem (GPT), $\text{card}(C \setminus f(D)) \leq 1$. This is contrary to the statement that $|f(z)| \geq \alpha - 1 > 0$. Assume that $\alpha = 1$. If $\beta = \gamma = 1$ then there exist sequences $r_n \searrow 0$ and $R_n \nearrow +\infty$ such that $f(r_n) \rightarrow 1$ and $f(R_n) \rightarrow 1$. Since $\lim_{n \rightarrow \infty} f_2(r_n) = \lim_{n \rightarrow \infty} f_1(R_n) = 0$ then there exists $M > 0$ such that $|f_1(r_n)| \leq M$ and $|f_2(R_n)| \leq M$ for $n \in \mathbb{N}$. By (2) we have that the functions $f_1(1/z)$ and $f_2(z)$ are bounded uniformly on the each circle $|z| = 1/r_n$ or $|z| = R_n$, respectively.

By Liouville's theorem $f_1(z) \equiv 0$ and $f_2(z) \equiv a_0$ and $f(z) \equiv a_0$. This is also contrary to our general assumption. If $\beta = 1$ and $\gamma > 1$, then, as above, $f_1(z) \equiv 0$. Hence $f = f_2$ and $f(0) = f_2(0) = 1$, $|f(z)| \geq f(|z| - 1)0$ for $z \neq 0$. If f is a polynomial then $f(z) \equiv f(1)$. The function f can not be transcendental entire because $|f(z)| \geq \gamma - 1)0$ for $|z| \geq 1$, what is contrary to GPT. In the case $\beta > 1$, $\gamma = 1$, due to Liouville's theorem and GPT, we also get a contradiction. The proof of Lemma 2 is complete.

Let $b > 0$ and $f(b) = 1$. Consider the function $g(z) := f(bz)$. The function g is holomorphic in D , satisfies (2) and $g(1) = 1$. Let $g(z) = \sum_{n \in \mathbb{Z}} b_n z^n$, where $b_n = a_n b^n$. By (2) we get $g(e^{i\theta}) \in \mathbb{R}$ for every $\theta \in \mathbb{R}$. Thus we have $\sum_{n \in \mathbb{Z}} b_n e^{in\theta} = \sum_{n \in \mathbb{Z}} b_n e^{-in\theta}$ and we must have $b_n = b_{-n}$ for $n \in \mathbb{Z}$. Therefore $g(z) = b_0 + \sum_{n \in \mathbb{N}} b_n (z^n + z^{-n})$ for $z \in D$ and $g(z) = g(1/z)$.

Lemma 3. *There exists $p \in \mathbb{N}$ such that $g(z) = b_0 + \sum_{n=1}^p b_n (z^n + z^{-n})$ for $z \in D$.*

Proof. Suppose that $z = 0$ and $z = \infty$ are essential singularities of the function g . Let $h = g|_{\mathbb{R}_+}$, $\alpha > 1$ and $E = h^{-1}([1, \alpha])$. The set E is closed in $\mathbb{R}_+$. If E is compact then we have a contradiction to the GPT. If E is unbounded then, due to Liouville's theorem, $g = \text{const} = 1$ (considerations in both cases are similar to those in the proof of Lemma 2). In every case we get a contradiction.

Finally we prove that $g(z) = \frac{1}{2}(z^p + z^{-p}) = j(z^p)$. By Lemma 1 the function g is even or is odd. If g is even then we can consider the function $g_1(z) = g(z^{1/2})$. The function g_1 is analytic for $z \in D$, satisfies (2) and $g_1(1) = 1$. If g_1 is also even then we take $g_2(z) = g_1(z^{1/2})$ and the function g_2 has the same properties as function g_1 . If g_2 is even then we analogously define the function g_3 . We can repeat this procedure. In the end we get the odd function g_k such that $g(z) = g_k(z^{2^k})$. Thus we can assume that g is odd.

It is easy to verify that if the function g satisfies (2) and $g(1) = 1$ then the function $h(z) = 2g^2(z) - 1$ also satisfies (2) and $h(1) = 1$. The function h is even and if g is odd then the function $h(z^{1/2})$ also must be odd. We have

$$\begin{aligned} h(z) = 2 \sum_{\substack{n \leq p \\ n \equiv 1 \pmod{2}}} b_n^2 (z^{2n} + z^{-2n}) + 4 \sum_{\substack{n < m \leq p \\ n, m \equiv 1 \pmod{2}}} b_n b_m ((z^{n+m} + z^{-(n+m)}) \\ + (z^{m-n} + z^{-(m-n)})) + 2 \sum_{\substack{n \leq p \\ n \equiv 1 \pmod{2}}} b_n^2 - 1. \end{aligned}$$

Thus we obtain

$$\sum_{\substack{n < m \leq p \\ n, m \equiv 1 \pmod{2} \\ 4 | n+m}} b_n b_m (z^{n+m} + z^{-(n+m)}) + \sum_{\substack{n < m \leq p \\ n, m \equiv 1 \pmod{2} \\ 4 | m-n}} b_n b_m (z^{m-n} + z^{-(m-n)}) \equiv 0.$$

Let $p \equiv 1 \pmod{4}$. Since $p - 2 \equiv -1 \pmod{4}$ and $n + m < 2p - 2$ if $n < p - 2$, we get that $b_{p-2} = 0$. By induction we prove that $b_{p-2(2k-1)} = 0$ for $k = 1, 2, \dots, (p-1)/4$. For $k = 1$ we have $b_{p-2} = 0$. Assume that $b_{p-2(2k-1)} = 0$ for $k = 1, \dots, l < \frac{p-1}{4}$ and consider the number $N = 2p - 2(2l+1)$. We have $N > p$ and if $n + m = N$ then we must have $n, m \geq p - 2(2l+1)$ and $0 = \sum_{k=0}^{l+1} b_{p-4k} b_{p-4l-2+4k} = b_p b_{p-4l-2}$. Hence $b_{p-4l-2} = 0$. Because $4|p-1$ and $m-n < p-1$ if $m < p$, we have $b_1 = 0$. By induction we can also prove that $b_{4k-3} = 0$ for $k = 1, \dots, (p-1)/4$. If $p \equiv 3 \pmod{4}$ then analogous considerations show that $b_n = 0$ for $n \equiv 1 \pmod{2}$, $n < p$.

Therefore we get $g(z) = a_p(z^p + z^{-p})$ and since $g(1) = 1$, we must have $g(z) = \frac{1}{2}(z^p + z^{-p})$. If we take $a = 1/b$ then we obtain $f(z) = j((az)^p)$. This completes the proof.

3. PROOF OF THEOREM 2

By Theorem 1 we have two cases: (i) there exists $C_1 \geq 1$ such that $T_n \circ f(z) \equiv C_1$ or (ii) there exist $a > 0$, $q \in \mathbb{N}$ such that $T_n \circ f(z) = j((az)^q)$. Consider the first case. Since T_n is a polynomial then the function f must be equal to a constant C . If $C \in [-1, 1]$ then, by the well-known properties of Chebyshev polynomials, we get that $C_1 = 1$ and $C = \cos(2\pi k/n)$ for some $0 \leq k \leq [n/2]$. Suppose that $C \notin [-1, 1]$. Then we can write $C = j(h(C))$, where $h: \mathbb{C} \setminus [-1, 1] \rightarrow \mathbb{C} \setminus \overline{B}(\overline{B} = \{|z| \leq 1\})$ is the inverse function to j , given by the formula $h(z) = z + \sqrt{z^2 - 1}$. Hence, by (3), $C_1 = j(h^n(C))$, $h^n(C) = h(C_1)$, $h(C) = \varepsilon |h(C_1)|$ and finally $C = j(\varepsilon C_2)$ where $\varepsilon^n = 1$ and $C_2 = |h(C_1)| > 1$. Now, assume that f is not equal to a constant. As above, we have $f(z) = j(h(f(z)))$ for $z \in D_1 = D \setminus f^{-1}([-1, 1])$. Thus we obtain $j(h^n(f(z))) = j((az)^q)$ for $z \in D_1$, $f^{-1}([-1, 1]) \subset S(1/a)$ and $h^n(f(z)) = (az)^q$ for $z \in D_2 = D_1 \cap (\mathbb{C} \setminus \overline{B}(1/a))$, where $\overline{B}(1/a)$, $S(1/a)$ denote the ball and the circle with centre at zero and radius $1/a$. Since the domain $D_3 = D_2 \setminus (-\infty, 0)$ is simply connected, then there exists in D_3 the principal branch of the function $(az)^{q/n}$, which we will also denote by $(az)^{q/n}$. Next, we can write $h^n(f(z)) = ((az)^{q/n})^n$ and we deduce that there exists ε , an n -th root of 1, such that $h(f(z)) = \varepsilon (az)^{q/n}$ for $z \in D_3$ and $f(z) = j(\varepsilon (az)^{q/n})$ for $z \in D_3$. Using analytic continuation we extend this equality to the domain $D \setminus (-\infty, 0)$. Finally, observe that since f is holomorphic in D , then we must have $q|n$. If we put $p = q/n$ then we complete the proof.

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