COLLISIONS OF LÉVY PROCESSES

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ABSTRACT. Let X, Y be two independent Lévy processes in R . We describe simple conditions on the density functions of X, Y which guarantee that the paths $X(\cdot), Y(\cdot)$ will have uncountably many collisions almost surely.

1. INTRODUCTION

Let $X(t)$ and $Y(t)$, $0 \leq t < \infty$, be two independent Lévy processes in R . The purpose of this note is to prove simple conditions on the density functions of $X(t)$ and $Y(t)$ which guarantee that $X(t), Y(t)$ almost surely have uncountably many collisions, i.e., $X(t) = Y(t)$ for uncountably many t . The problem stems from a work of Pólya [5] on random walks. In case $X(t)$ and $Y(t)$ are stable processes, the problem was investigated by Jain and Pruitt [2]. We assume that our Lévy processes are right continuous and have left hand limits everywhere. Our result is

Theorem 1. Let $X(t)$, $0 \leq t < \infty$, be a Lévy process in R , defined on some probability space $(\Omega, \mathcal{F}, P)$. Assume that each $X(t)$, $t > 0$, has a bounded continuous density function $p(t, x)$. Assume further that $p(t, x)$ is everywhere positive and for some $T > 0$

$$(1.1) \quad \int_0^T \|p(t, \cdot)\|_{\infty} dt < \infty.$$

Let $Y(t)$, $0 \leq t < \infty$, be another Lévy process, independent on $X(t)$ and defined also on $(\Omega, \mathcal{F}, P)$. Assume that each $Y(t)$, $t > 0$, has a continuous density function $q(t, x)$. Then, for almost all ω , $X(t, \omega) = Y(t, \omega)$ for uncountably many t .

Remark. Let $\psi(y)$ be the characteristic exponent of $X(t)$, i.e., $Ee^{iyX(t)} = e^{-t\psi(y)}$. Note that (1.1) holds when

$$\int_{-\infty}^{\infty} \frac{1 - e^{-\operatorname{Re} \psi(y)}}{\operatorname{Re} \psi(y)} dy < \infty.$$

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A particular case is when the Blumenthal-Gettoor [1] lower index β'' of $X(t)$ is greater than 1.

In contrast to Jain and Pruitt [2], where potential-theoretic tools were used, our proof of Theorem 1 is based on a result of Marcus [4] on the capacity of level sets of stochastic processes, and is related to the techniques used in LeGall, Rosen and Shieh [3].

2. THE PROOF OF THEOREM 1

The proof is an application of Marcus [4, Theorem 1] to $Z = X - Y$, with $u = 0$ there, on a sequence of time intervals. We divide the whole proof into several steps.

Step 1. Define $Z(t) = X(t) - Y(t)$, $t \in [0, T]$, where T is given in (1.1). Let $r(s; x)$ and $r(s, t; x, y)$ denote respectively the density function of $Z(s)$ and $(Z(s), Z(t))$. Using the Markov property and the independence of X , Y , we see that, for $0 < s < t \leq T$,

$$\begin{aligned} r(s; x) &= \int_z q(s, z - y_0) p(s, x + z - x_0) dz, \\ r(s, t; x, y) &= \int_z \int_{z'} q(s, z - y_0) q(t - s, z' - z) p(s, x + z - x_0) \\ &\quad \cdot p(t - s, y - x + z' - z) dz dz', \end{aligned}$$

where $x_0 = X(0)$, $y_0 = Y(0)$. Hence, by our assumption on p and q , both $r(s; \cdot)$ and $r(s, t; \cdot, \cdot)$ are continuous, and $r(s, \cdot)$ is everywhere positive. Note that $q(t, x)$ must be positive on a set of positive Lebesgue measure.

Step 2. Observe that

$$\sup_x r(s; x) \leq h(s) \equiv \|p(s, \cdot)\|_\infty,$$

and $\sup_{x, y} r(s, t; x, y) \leq g(s, t) \equiv \|p(s, \cdot)\|_\infty \|p(t - s, \cdot)\|_\infty$. By our assumption on p , both $h(s)$ and $g(s, t)$ are integrable.

Step 3. Since

$$\int_0^T \|p(t, \cdot)\|_\infty dt < \infty,$$

we have

$$\sum_{m=1}^{\infty} \int_0^T \|p(t, \cdot)\|_\infty 1_{\{T/2^m < t \leq T/2^{m-1}\}} dt < \infty.$$

Thus, we may find an increasing sequence of positive numbers $\{a_m : m = 1, 2, \dots\}$ such that $\lim_{m \rightarrow \infty} a_m = \infty$ and

$$\sum_{m=1}^{\infty} a_m \int_0^T \|p(t, \cdot)\|_\infty 1_{\{T/2^m < t \leq T/2^{m-1}\}} dt < \infty.$$

We take ϕ such that $\phi(2^{-m}) = a_m$ for m large. We then extend ϕ to be a decreasing continuous function $\phi : (0, \infty) \rightarrow [1, \infty]$ with $\phi(0+) = \infty$. By the

above display,

$$\int_0^T \|p(t, \cdot)\|_\infty \phi(t) dt < \infty.$$

Thus,

$$\begin{aligned} \int_{s=0}^T \int_{t=s}^T r(s, t; 0, 0) \phi(t-s) dt ds \\ \leq \int_{s=0}^T \int_{t=s}^T \|p(s, \cdot)\|_\infty \|p(t-s, \cdot)\|_\infty \phi(t-s) dt ds < \infty. \end{aligned}$$

Step 4. Denote by

$$G(t, \omega) = \{t \in [0, T] : Z(t, \omega) = 0\}$$

the level set $Z(t, \omega) = 0$ on $[0, T]$. Denote the fact that the capacity of a set $A \subset [0, T]$ is strictly positive with respect to ϕ by $\text{Cap}_\phi(A) > 0$. We observe that

$$\begin{aligned} P(X(t, \omega) = Y(t, \omega) \text{ for uncountably many } t \in [0, T]) \\ \geq P(\text{Cap}_\phi G(T, \omega) > 0). \end{aligned}$$

Now, we show how to apply Marcus [4, Theorem 1] to conclude that

$$P(\text{Cap}_\phi G(T, \omega) > 0) \geq \delta_T > 0,$$

where

$$\delta_T = \frac{\left(\int_0^T r(s; 0) ds\right)^2}{\int_0^T \int_0^T r(s, t; 0, 0) ds dt}.$$

In fact, Steps 1, 2 and 3 assert that the conditions in [4, Theorem 1] are satisfied, with $u = 0$ there. However, our $Z(\cdot, \omega)$ may have countably many discontinuities, while Marcus has assumed continuity of the paths. We remark that his assumption can be weakened, by observing the proof in [4, pp. 281–283]. The point is that the measure $Y(\cdot, \omega)$ constructed on p. 281 is still supported by $G(T, \omega)$, even if his $X(\cdot, \omega)$ has countably many discontinuities (cf. p. 282, line 5). We reason as follows. Since $Y(\cdot, \omega)$ satisfies (see p. 282)

$$\int_{t=0}^T \int_{s=0}^T \phi(|t-s|) dY(s, \omega) dY(t, \omega) < \infty,$$

and $\phi(0+) = \infty$, $Y(\cdot, \omega)$ has no atoms. Since $Y(\cdot, \omega)$ is atomless and $X(\cdot, \omega)$ has at most countably many discontinuities, to show that $Y(\cdot, \omega)$ is supported by $G(T, \omega)$ it suffices to see that $Y(A(\omega), \omega) = 0$, where

$$A(\omega) = \{t \in [0, T] : X(\cdot, \omega) \text{ is continuous at } t \text{ and } X(t, \omega) \neq 0\}.$$

For $t \in A(\omega)$, there exist $\varepsilon > 0$ and $\delta > 0$ such that $|X(s, \omega) - 0| > \varepsilon$ for all $s : |s-t| < 2\delta$. Now, for all rationals $s_1, s_2 : t-2\delta < s_1 < t-\delta < t < t+\delta < s_2 <$

$t + 2\delta$, by the definition of $Y(\cdot, \omega)$ in p. 281, we have $Y(s_2, \omega) - Y(s_1, \omega) = 0$. Thus, $Y((t - \delta, t + \delta), \omega) = 0$, and hence $Y(A(\omega), \omega) = 0$. We have therefore shown that Marcus' result is applicable. The lower bound δ_T comes from [4, p. 282, (2.10)].

Step 5. Define, for $n = 0, 1, 2, \dots$,

$$A_n = \{\omega : X(t, \omega) = Y(t, \omega) \text{ for uncountably many } t, t \in [nT, (n+1)T]\}.$$

Then $\{A_n\}$ are independent, and each $P(A_n) \geq \delta_T$. Thus, by Borel-Cantelli Lemma, $P(A_n \text{ i.o.}) = 1$. This proves Theorem 1.

3. CONCLUDING REMARK

The following question is natural: To what extent can Theorem 1 be extended to Markov processes which are not Lévy? An examination on the proof of Theorem 1 will show that, if we assume that X is time-homogeneous Markovian, with continuous transition density functions $p(t, x, y)$, and that

$$\int_0^T \|p(t, \cdot, \cdot)\|_\infty dt < \infty,$$

then the Steps 1, 2, 3, 4 will be all right. As for the Step 5, we may apply conditional Borel-Cantelli Lemma, under certain ergodicity assumption that $p(s, \cdot, \cdot) \geq \beta(s) > 0$. Since all the things are not far away from Theorem 1 and its proof, we leave it for the reader to formulate the relevant reasoning and result.

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