AUTOMORPHIC-DIFFERENTIAL IDENTITIES IN RINGS

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(Communicated by Donald S. Passman)

Dedicated to the memory of my teacher and friend, I. N. Herstein

ABSTRACT. Let R be a ring and f an endomorphism obtained from sums and compositions of left multiplications, right multiplications, automorphisms, and derivations. We prove several results relating the behavior of f on certain subsets of R to its behavior on all of R . For example, we prove (1) if R is prime with ideal $I \neq 0$ such that $f(I) = 0$, then $f(R) = 0$, (2) if R is a domain with right ideal $\lambda \neq 0$ such that $f(\lambda) = 0$, then $f(R) = 0$, and (3) if R is prime and $f(\lambda^n) = 0$, for λ a right ideal and $n \geq 1$, then $f(\lambda) = 0$. We also prove some generalizations of these results for semiprime rings and rings with no non-zero nilpotent elements.

1. INTRODUCTION AND NOTATION

In this paper we will deal with endomorphisms of rings which arise from taking sums and compositions of left multiplications, right multiplications, automorphisms, and derivations. We will refer to them as *automorphic-differential endomorphisms*. We will examine some questions on automorphic-differential endomorphisms which arose quite naturally in the study of the invariants of group actions and Lie algebra actions. When examining the action of a group or Lie algebra one would like to explicitly produce non-zero invariants inside of certain ideals or subrings. This is often done with a trace map, which maps the ring into the subring of invariants. For group actions, this is done by applying sums of automorphisms and for Lie algebras, by applying sums and compositions of derivations. In these cases the trace map is an automorphic-differential endomorphism and one often wants to determine if a trace map which is non-zero on a ring is non-zero on a particular ideal or subring.

There are already some results along these lines. Montgomery [5] shows that if a finite group acts on a domain then the trace map is non-zero if and only if it is non-zero on every non-zero one-sided ideal. Chung and Luh [2] show that

Received by the editors April 14, 1988. The contents of this paper were presented at the American Mathematical Society Meeting at Kent State University, April 3, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 16A12, 16A72.

Research supported, in part, by a grant from the Faculty Research and Development Fund of the College of Liberal Arts and Sciences at DePaul University.

if d is a derivation of a prime ring then d^n is zero on a non-zero ideal if and only if d^n is zero on the entire ring. Lanski [3] extends this result to show that if a derivation is algebraic on a non-zero ideal of a prime ring then the same polynomial in the derivation vanishes on the entire ring. Leroy and Matczuk [4] also prove this result as well as the analogous result for an automorphism. Although these results deal only with a finite group or a single automorphism or derivation they all suggest that an automorphic-differential endomorphism which vanishes on a non-zero ideal of a prime ring or on a non-zero one-sided ideal of a domain should vanish on the entire ring. We call an automorphic-differential endomorphism which vanishes on a ring an *automorphic-differential identity*.

We will generalize the above results by proving

Theorem A. *Let R be a prime ring, $I \neq 0$ an ideal, and f an automorphic-differential endomorphism. If $f(I) = 0$ then $f(R) = 0$.*

Theorem B. *Let R be a domain, $\lambda \neq 0$ a right ideal, and f an automorphic-differential endomorphism. If $f(\lambda) = 0$ then $f(R) = 0$.*

In prime rings an automorphic-differential endomorphism which vanishes on a right ideal need not be an automorphic-differential identity. However, we can show that an automorphic-differential endomorphism which vanishes on a power of a right ideal of a prime ring vanishes on the ideal. We will prove

Theorem C. *Let R be a prime ring, λ a right ideal of R , and f an automorphic-differential endomorphism. If $f(\lambda^n) = 0$, for $n \geq 1$, then $f(\lambda) = 0$.*

We will also extend various aspects of Theorems A, B, C to semiprime rings and to rings with no non-zero nilpotent elements.

Despite generalizing results which require either difficult combinatorial or structure theoretic proofs, our results will be obtained through fairly elementary means. We will embed our automorphic-differential endomorphisms in a set of polynomials which resembles a skew group ring and a Lie algebra smash product [1]. We avoid the calculations required for a fixed endomorphism [2,3] by instead exploiting the properties of endomorphisms with minimal degrees as polynomials.

The techniques used in this paper can also be used to prove results on Lie ideals, anti-automorphisms, and rings whose invariants are central. However they require far more technical arguments involving prime rings satisfying polynomial identities and generalized polynomial identities. To include these more technical results in this paper, we feel would obscure the ideas which enable us to prove Theorems A, B, and C as well as other results on prime and semiprime rings. Our goal is for this paper to be self-contained and to return to those other questions in a more technical paper.

Suppose now that f is an automorphic-differential endomorphism of ring R ; we will first show that we can write $f = f_1 + \cdots + f_n$ where each f_i is of

the form $L_a g d_1 \dots d_m$ where L_a is left multiplication by the element a ; g is an automorphism; and $d_1, \dots, d_m$ are derivations.

If $b \in R$, define the maps $L_b(r) = br$, $T_b(r) = rb$, and $d_b(r) = rb - br$, for all $r \in R$. Since $d_b = T_b - L_b$, we have $T_b = d_b + L_b$, therefore in f we may replace every right multiplication by the sum of a derivation and a left multiplication. Now suppose $c, r, s \in R$ and let d be a derivation and g an automorphism; then we have

- (1) $dL_c(r) = d(cr) = cd(r) + d(c)r = (L_c d + L_{d(c)})(r)$
- (2) $gL_c(r) = g(cr) = g(c)g(r) = L_{g(c)}g(r)$
- (3) $g^{-1}dg(rs) = g^{-1}d(g(r)g(s)) = g^{-1}dg(r)s + rg^{-1}dg(s)$.

By (1) and (2), in f we may apply all our left multiplications after all the derivations and automorphisms are applied. Furthermore, by (3), $g^{-1}dg$ is a derivation δ , thus $dg = g\delta$ and so, if f we may apply all the derivations before we apply the automorphisms. Therefore f can now be written as a sum of endomorphisms, each of which is a composition of derivations, followed by automorphisms, followed by left multiplications. However the composition of automorphisms is again an automorphism and the composition of left multiplications is again a left multiplication. Thus we write $f = f_1 + \dots + f_n$, where each f_i is of the form $L_a g d_1 \dots d_m$.

We can now fix the notation that we will use throughout this paper. Let R be a ring with a set of derivations $\{d_1, d_2, \dots, d_l\}$ and a set of automorphisms $\{g_1, g_2, \dots, g_k\}$. The derivations and automorphisms are not required to be distinct endomorphisms. We let $X = \{x_1, x_2, \dots, x_l\}$ and $Y = \{y_1, y_2, \dots, y_k\}$ be sets of variables with each x_i corresponding to the derivation d_i and each y_j corresponding to the automorphism g_j . We define $R[X, Y]$ to be the free left R -module with basis consisting of all monomials of the form $y_i \Delta_i$, where $y_i \in Y$ and Δ_i is a product of elements from X . In forming Δ_i , the elements of X are considered to be free non-commuting variables and we allow Δ_i to be an empty product. We call $R[X, Y]$ a set of *automorphic-differential polynomials* and we define a left action of the set $R[X, Y]$ on the ring R . It will certainly suffice to define our action of $R[X, Y]$ in terms of monomials and then extend additively. For $a \in R$, $y_j \in Y$, and $x_{i_1}, \dots, x_{i_r} \in X$ we define the action $ay_j x_{i_1} \dots x_{i_r} \cdot r = L_a g_j \cdot d_{i_1} \dots d_{i_r}(r)$, for all $r \in R$. In general, if $w \in R[X, Y]$ and $A \subset R$ we will denote the image of A under the action of w as either $w \cdot A$ or $w(A)$, depending upon the context.

If $w \in R[X, Y]$, by the *degree* of w we mean its degree when viewed solely as a polynomial in the variables $x_i \in X$. If $w \in R[X, Y]$ has degree $n \geq 0$ its *length* will be the number of degree n monomials in w . For example, $ay_1 x_3 x_2 + bx_3 x_2 + cy_2 x_4 \in R[X, Y]$ has degree 2 and length 2.

It will also be important for $R[X, Y]$ to have the structure of an R -bimodule. In defining the right action of R on $R[X, Y]$ we must be consistent with the

left action of $R[X, Y]$ on R . Therefore, as in formulas (1) and (2), we let

$$x_i c = c x_i + d_i(c) \quad \text{and} \quad y_i c = g_i(c) y_i \quad \text{for } c \in R, \quad x_i \in X, \quad y_i \in Y.$$

It is now easy to see that, for $s \in R$, we have $x_i c \cdot s = x_i \cdot (cs)$ and $y_i c \cdot s = y_i \cdot (cs)$ and so the right action of R on $R[X, Y]$ is consistent with the left action of $R[X, Y]$ on R .

The final piece of notation we will need is if $A \subset R$ we let $l(A)$ and $r(A)$ denote, respectively, the left and right annihilators of A . More precisely, $l(A) = \{r \in R \mid ra = 0, \text{ for all } a \in A\}$ and $r(A) = \{r \in R \mid ar = 0, \text{ for all } a \in A\}$.

From the preceding discussion it is now clear that if f is any automorphic-differential endomorphism then there exist finite sets X, Y corresponding, respectively, to the derivations and automorphisms in f such that the action of f on R can be induced by at least one element in $R[X, Y]$. In the next section we will obtain the desired information about f by exploiting the bimodule structure of $R[X, Y]$.

2. THE MAIN RESULTS

All of our results on automorphic-differential endomorphisms will follow from

Proposition. *Let R be any ring, $R[X, Y]$ a set of automorphic-differential polynomials, λ a right ideal of R , and P a prime ideal of R . Suppose $N(P) = \{w \in R[X, Y] \mid w(\lambda) \subset P\}$ and $M(P) = \{w \in R[X, Y] \mid w(R) \subset P\}$; then either $N(P) = M(P)$ or $ag(\lambda) \subset P$ for some $a \in R$, $a \notin P$ and an automorphism g corresponding to an element in Y .*

Proof. Suppose $N(P) \neq M(P)$; since $M(P) \subset N(P)$ there exist elements in $N(P)$ not in $M(P)$. From among elements in $N(P)$ not in $M(P)$, let n be the smallest degree and m the shortest length from among those degree n elements. Now suppose $w \in N(P)$, $w \notin M(P)$ has degree n and length m ; then $w = ay\Delta + (m-1 \text{ monomials of degree } n) + (\text{monomials of degree } < n)$ where $a \in R$, $y \in Y$, and Δ is a degree n monomial from elements in X . If $a \in P$ then $ay\Delta \cdot R \subset P$, thus $(w - ay\Delta) \cdot \lambda \subset P$ and $w - ay\Delta$ has either smaller degree or shorter length than w . Therefore $w - ay\Delta \in M(P)$, hence $w = (w - ay\Delta) + ay\Delta \in M(P)$, a contradiction. Thus $a \notin P$.

If g is the automorphism corresponding to y , let $r \in g(\lambda)$ and consider $arw - wg^{-1}(r)g^{-1}(a)$; since $g^{-1}(r) \in \lambda$ we have $wg^{-1}(r)g^{-1}(a) \cdot R = w \cdot g^{-1}(r)g^{-1}(a)R \subset w \cdot \lambda \subset P$. Therefore $wg^{-1}(r)g^{-1}(a) \in M(P)$ and neither arw nor $wg^{-1}(r)g^{-1}(a)$ contains any degree n monomials not already present in w . Now $arw - wg^{-1}(r)g^{-1}(a) = (ara - ara)y\Delta + (\text{at most } m-1 \text{ degree } n \text{ monomials}) + (\text{monomials of degree } < n)$; thus $arw - wg^{-1}(r)g^{-1}(a) \in M(P)$ as it is an element of $N(P)$ of either smaller degree or shorter length than w . Since $arw - wg^{-1}(r)g^{-1}(a)$, $wg^{-1}(r)g^{-1}(a) \in M(P)$ we have $arw \in M(P)$, hence $ar(w \cdot R) = arw \cdot R \subset P$, for any $r \in g(\lambda)$. Therefore $ag(\lambda)w(R) \subset P$

and combining the facts that $ag(\lambda)$ is a right ideal, P is a prime ideal, and $w(R) \not\subset P$ we obtain the conclusion $ag(\lambda) \subset P$.

We can now prove our first main result from which Theorems A and B will follow as special cases.

Theorem 1. *Let R be a prime ring, f an automorphic-differential endomorphism, and λ a right ideal with $l(\lambda) = 0$. If $f(\lambda) = 0$ then $f(R) = 0$.*

Proof. Let X and Y be finite sets such that f is induced by some $w \in R[X, Y]$. For any automorphism g of R , $l(g(\lambda)) = l(\lambda) = 0$, therefore $ag(\lambda) \neq 0$ for any $0 \neq a \in R$. Now apply the Proposition with the prime $P = 0$, since $w \in N(P)$ and $ag(\lambda) \not\subset P$, we have $w \in M(P)$, thus $f(R) = 0$.

If $\lambda \neq 0$ is either a right ideal in a domain or an ideal in a prime ring then $l(\lambda) = 0$. In light of Theorem 1, we immediately obtain Theorems A and B.

Theorem A. *Let R be a prime ring, $I \neq 0$ an ideal, and f an automorphic-differential endomorphism. If $f(I) = 0$ then $f(R) = 0$.*

Theorem B. *Let R be a domain, $\lambda \neq 0$ a right ideal, and f an automorphic-differential endomorphism. If $f(\lambda) = 0$ then $f(R) = 0$.*

We would like to generalize these results to semiprime rings and rings with no non-zero nilpotent elements. However, in such rings if an automorphic-differential endomorphism vanishes on a non-zero ideal it need not be an automorphic-differential identity. This is easy to see in the following example: let R_1, R_2 be domains with $d \neq 0$ a derivation of R_1 . Then $R_1 \oplus R_2$ has no non-zero nilpotent elements and $d \oplus 0$ is a non-zero derivation vanishing on the ideal $0 \oplus R_2$. In this example, $l(0 \oplus R_2) \neq 0$; if we assume, as in Theorem 1 that our ideal has zero left annihilator, we can prove

Theorem 2. *Let R be a semiprime ring, f an automorphic-differential endomorphism, and I an ideal with $l(I) = 0$. If $f(I) = 0$ then $f(R) = 0$.*

Proof. Let X, Y be finite sets such that f is induced by some $w \in R[X, Y]$. If P is a prime ideal of R , by the Proposition, either $N(P) = M(P)$ or $ag(I) \subset P$ where $a \notin P$ and g correspond to an element in Y . Since P is prime and $g(I)$ is an ideal, if $ag(I) \subset P$ then $g(I) \subset P$; hence, for any prime P , either $w(R) \subset P$ or $g(I) \subset P$. If $g_1, \dots, g_k$ are all the automorphisms corresponding to the set Y then $w(R)g_1(I)g_2(I)\dots g_k(I) \subset P$, for every prime P . However, since R is semiprime the intersection of its primes is 0, hence $w(R)g_1(I)g_2(I)\dots g_k(I) = 0$. Each g_i is an automorphism, thus $l(g_i(I)) = l(I) = 0$, hence $w(R) = 0$. Thus $f(R) = 0$.

For rings with no non-zero nilpotent elements we can extend Theorem 2 to right ideals.

Theorem 3. *Let R be a ring with no non-zero nilpotent elements, f an automorphic-differential endomorphism, and λ a right ideal with $l(\lambda) = 0$. If $f(\lambda) = 0$ then $f(R) = 0$.*

Proof. Let P be a completely prime ideal of R , that is, a prime such that R/P is a domain. We proceed as in the proof of Theorem 2; if $w \in R[X, Y]$ induces f then either $w(R) \subset P$ or $ag(\lambda) \subset P$. Since $a \notin P$ and P is completely prime, we have $w(R) \subset P$ or $g(\lambda) \subset P$ and so, $w(R)g_1(\lambda) \dots g_k(\lambda)$ is contained in every completely prime ideal. However, in a ring with no non-zero nilpotent elements the intersection of the completely prime ideals is 0, therefore we conclude $w(R) = 0$, hence $f(R) = 0$.

Neither Theorem 2 nor Theorem 3 completely generalizes Theorem 1 to semiprime rings. We suspect that the following generalization is true, but we have been unable to prove it:

Question 1. *Suppose R is a semiprime ring, f an automorphic-differential endomorphism, and λ a right ideal with $l(\lambda) = 0$. If $f(\lambda) = 0$, is $f(R) = 0$?*

It is easy to see that for an arbitrary right ideal λ of a prime ring, an automorphic-differential endomorphism f can vanish on λ without being an automorphic-differential identity. However, we will see that if f vanishes on a power of λ , then it must vanish on λ . This will follow fairly directly from Theorem A.

Theorem C. *Let R be a prime ring, f an automorphic-differential endomorphism, and λ a right ideal. If $f(\lambda^n) = 0$, for $n \geq 1$, then $f(\lambda) = 0$.*

Proof. We may, without loss, first assume that R contains a unit element. Furthermore, by induction, we may also assume that $n = 2$. Note that $\lambda^2 = (\lambda)(\lambda) \supset (\lambda R)\lambda = \lambda(R\lambda)$ and let $I = R\lambda$. Suppose $w \in R[X, Y]$ induces f and let $b \in \lambda$; therefore $wb \in R[X, Y]$ and $wb \cdot I = w \cdot (bI) \subset w \cdot (\lambda I) \subset w \cdot (\lambda^2) = 0$. Applying Theorem A to wb , we see that $(wb) \cdot R = 0$. Hence $w(b) = (wb) \cdot 1 \in (wb) \cdot R = 0$, thus $f(b) = 0$, for all $b \in \lambda$. Therefore $f(\lambda) = 0$.

For the case where no automorphisms are present, we would like to extend Theorem C to semiprime rings. To do so, we first need the following lemma on the invariance of right annihilators:

Lemma. *Let R be a semiprime ring with ideal J and derivation d ; then $d(r(J)) \subset r(J)$.*

Proof. Since $Jr(J) = 0$ and $d(J^2) \subset J$, we have $0 = d(J^2r(J)) = J^2d(r(J)) \supset (Jd(r(J)))^2$. By the semiprimeness of R , $Jd(r(J)) = 0$, hence $d(r(J)) \subset r(J)$.

We now extend Theorem C, in the absence of automorphisms, to semiprime rings. In this case, we will refer to our endomorphism as a *differential endomorphism*. Later we will deal with endomorphisms involving automorphisms, but not derivations and we will refer to them as *automorphic endomorphisms*.

Theorem 4. *Let R be a semiprime ring, f a differential endomorphism, and λ a right ideal. If $f(\lambda^n) = 0$, for $n \geq 1$, then $f(\lambda) = 0$.*

Proof. As in the proof of Theorem C, we assume that R has a unit, $n = 2$, and let $I = R\lambda$ while noting that $\lambda I \subset \lambda^2$. Suppose $w \in R[X]$ induces f and let $b \in \lambda$; we want to show $w(b) = 0$. If P is a prime ideal of R , we have $(wb) \cdot I = w \cdot (bI) \subset w \cdot \lambda^2 = 0 \subset P$, therefore by the Proposition, either $(wb) \cdot R \subset P$ or $aI \subset P$, for some $a \notin P$. Hence either $w(b) \in P$ or $I \subset P$, thus we conclude $R\lambda w(b) = Iw(b) \subset P$ and so, $\lambda w(b) \subset P$. Since the intersection of the primes of R is 0, we have $\lambda w(b) = 0$.

By definition, $\lambda r(\lambda) = 0$ therefore $(r(\lambda)\lambda)^2 = r(\lambda)\lambda r(\lambda) = 0$, hence $r(\lambda)\lambda = 0$. Therefore $\lambda \subset r(r(\lambda))$ and, by repeated use of the previous lemma with $J = r(\lambda)$, we have $w(\lambda) \subset w \cdot r(r(\lambda)) \subset r(r(\lambda))$. However, by the previous paragraph, $w(b) \in r(\lambda)$, therefore $w(b) \in r(\lambda) \cap r(r(\lambda))$. Since $r(\lambda) \cap r(r(\lambda))$ is of square zero, $r(\lambda) \cap r(r(\lambda)) = 0$ and so, $w(b) = 0$.

A well-known open question in the study of derivations acting on semiprime rings is

Question 2. *Are the minimal primes of a semiprime ring invariant under all derivations?*

For an arbitrary ring the minimal primes need not be invariant; however for algebras over the rationals the minimal primes are invariant, even if the algebra is not semiprime [6]. A positive answer to Question 2 would enable us to extend Theorem C to semiprime rings. This is because we can prove a semiprime version of Theorem C when the minimal primes are invariant under the derivations in our automorphic-differential endomorphism. More precisely, we prove

Theorem 5. *Let R be a semiprime ring, f an automorphic-differential endomorphism, and λ a right ideal. If the minimal primes of R are invariant under the derivations in f and if $f(\lambda^n) = 0$, for $n \geq 1$, then $f(\lambda) = 0$.*

Proof. This proof will be somewhat similar to the proof of our Proposition, however we will need to make some changes in the way we represent elements of $R[X, Y]$. Whereas we have been viewing elements of $R[X, Y]$ as sums of monomials of the form $ay\Delta$, we can instead use formulas (1), (2), (3) from Section 1 to view the monomials in $R[X, Y]$ as being of the form Δay . Viewing elements of $R[X, Y]$ in this way, we can still refer to the degree and length of elements in $R[X, Y]$.

Let P be a minimal prime of R and suppose there exist $w \in R[X, Y]$ such that $w(\lambda^2) \subset P$ and $w(\lambda) \not\subset P$. Let w have minimal degree n and shortest length m from among such elements; thus $w = \Delta ay + (m - 1 \text{ monomials of degree } n) + (\text{monomials of degree } < n)$. For $r \in g(\lambda)$ consider $arw - wg^{-1}(r)g^{-1}(a) = \Delta(ara - ara)y + (\text{at most } m - 1 \text{ monomials of degree } n) + (\text{monomials of degree } < n)$.

Since $arw \cdot \lambda^2 \subset P$ and $wg^{-1}(r)g^{-1}(a) \cdot \lambda \subset w \cdot g^{-1}(r)g^{-1}(a)\lambda \subset w \cdot \lambda^2 \subset P$, we conclude, by the minimality of w , that $(arw - wg^{-1}(r)g^{-1}(a)) \cdot \lambda \subset P$, hence $arw \cdot \lambda \subset P$. Therefore $arw(\lambda) \subset P$, for all $r \in g(\lambda)$, thus $ag(\lambda)w(\lambda) \subset P$ and so, $ag(\lambda) \subset P$. However, in this case $\Delta ay \cdot \lambda = \Delta \cdot ag(\lambda) \subset \Delta \cdot P \subset P$ thus, by the minimality of w , $(w - \Delta ay) \cdot \lambda \subset P$. Therefore $w(\lambda) \subset (w - \Delta ay) \cdot \lambda + \Delta ay \cdot \lambda \subset P$, a contradiction. As a result, if $w \in R[X, Y]$ such that $w(\lambda^2) \subset P$, it follows that $w(\lambda) \subset P$.

Therefore if $f(\lambda^2) = 0$, we have $f(\lambda) \subset P$, for every minimal prime P . Since the intersection of the minimal primes of R is 0, we conclude that $f(\lambda) = 0$.

If no derivations are present in an automorphic-differential endomorphism, then certainly the hypothesis on the invariance of minimal primes in Theorem 5 is satisfied. We then immediately have

Theorem 6. *Let R be a semiprime ring, f an automorphic endomorphism, and λ a right ideal. If $f(\lambda^n) = 0$, for $n \geq 1$, then $f(\lambda) = 0$.*

We suspect that the conclusion of Theorem 5 is still true, even if the minimal primes of R fail to be invariant. Since we have been unable to prove this we leave it as

Question 3. *Suppose R is a semiprime ring, f an automorphic-differential endomorphism, and λ a right ideal. If $f(\lambda^n) = 0$, for $n \geq 1$, must $f(\lambda) = 0$?*

We conclude this paper by looking at our results on automorphic-differential endomorphisms in a different context. If a Lie algebra L acts on a ring R as derivations then this extends to an action of the universal enveloping algebra $U(L)$ by composition of derivations and finally to an action of $R \# U(L)$ on R by composition of derivations and left multiplications. Similarly, if a group G acts on R as automorphisms this extends to an action of the skew group ring $R * G$. But $R \# U(L)$ and $R * G$ are examples of Hopf algebra smash products $R \# H$ and we can view our results in this context. For example, as a special case of Theorem A we have the following:

Theorem. *Let R be a prime ring and H either an enveloping algebra or group algebra acting on R . If $f \in R \# H$ such that $f(I) = 0$, for $I \neq 0$ an ideal of R , then $f(R) = 0$.*

Similarly, all the results in this paper have, as special cases, restatements in terms of Hopf algebra actions. It is natural to wonder if the results in this paper extend to arbitrary Hopf algebra actions. This leads to the following general question:

Question 4. *Suppose R is a ring, H a Hopf algebra acting on R , and let $f \in R \# H$. If R is special, such as being prime or a domain, and if $\lambda \neq 0$ is a special subset of R , such as being an ideal or a one-sided ideal, must $f(R) = 0$ whenever $f(\lambda) = 0$?*

The key to the proofs in this paper was our ability to use minimal degree and shortest length arguments. These arguments can be used for enveloping algebras and group algebras, but it is not clear how to proceed for arbitrary Hopf algebras.

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