“PLACE FUNCTIONS”: ALIAS CONTINUOUS FUNCTIONS ON THE STONE SPACE

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ABSTRACT. Caratheodory's system of place functions on a Boolean σ -ring is shown to be canonically isomorphic—order and algebraically—to the system of continuous extended real-valued functions over its Stone space. This enables one to dispense with the imperspicuous formulae and calculations of [CL] in favor of the more straightforward and familiar ones of function spaces; and to subsume the proposed generalization of integration under standard treatment.

A “nest of somas” [CL §69] is an isotone map from the reals to a Boolean ring (fixed in the sequel). It is pointed out in [CL §69] how “finitely-valued place functions”—i.e. extended real-valued step functions ring—determine such nests from which the functions may be recovered. Here, this will be extended from the step functions to the continuous functions on the Stone space of the ring; and it will be pointed out that the most general nest—hence the most general “place function”—will be obtained by this procedure, resulting in the advantages pointed out above.

The Stone space of a Boolean ring is the set of 0-preserving lattice morphisms to $\{0, 1\}$ —equivalently, its decompositions into a nonvoid ideal and a (possibly void) complementary filter. The $S(y)$ of any nest are divided by such a decomposition into a hereditary and a disjoint co-hereditary subset of the ring; on the real arguments of the nest this division effects a cut, which will be taken as the value at this Stone space point of the function which is to correspond to the nest. A ring element is represented in the Stone space by the subset of the decompositions for which it belongs to the filter: thus on the Stone space X , one has for corresponding entities: $\{x : f(x) < y\} \subset S(y)$ and dually, i.e. $S(y) \subset \{x : f(x) \leq y\}$.

The nests are pre-ordered as inverse to $S(y) \subset T(z)$ for $y < z$; this entails that the cut determined by T at each x is $\leq$ that determined by S , hence for the corresponding functions that $g \leq f$ on X ; and conversely (e.g. by the inclusions just set up). Now the place functions are defined as the equivalence classes of nests under the identification which makes this preorder a true partial

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order. Thus the poset of place functions on a Boolean ring is order-isomorphic to a poset of extended real-valued point functions on its Stone space.

It is also clear which point functions are images of place functions: they are just those for which the sets where $f(x) < y$ and $f(x) \leq y$ enclose a clopen set (for each y). Since $f(x) < z$ resp. $> z$, just when x belongs resp. fails to belong to one of these clopen sets for some $y < z$ resp. $> z$, these representing functions are continuous. To achieve conversely that every continuous function is a place function, we restrict henceforth (as in [CL]) to σ -rings: their Stone spaces are characterized [V p.42] as having (cl)open closures for open F_σ 's. Then the clopen closure of $\{x : f(x) < y\}$ is contained in the closed $\{x : f(x) \leq y\}$ thus showing the continuous f to be a place function. It is known that over Stone spaces of this kind the continuous extended real-valued functions form a σ -lattice (although the countably infinite extrema are not computed pointwise—see e.g. [V, V.1]; it must be admitted that the proof using the place function representation is simpler) hence this holds for their order-isomorphic place functions.

The infimum of the place function $S(y)$ on a soma S is defined [CL §77] as the cut determined by the (non-)disjointness of S and $S(y)$; and the supremum by the (non-)inclusion of S in $S(y)$. These could be defined for any subset S of the Stone space; for singletons each reduces to the above definition of the corresponding point function; and the location of $S(y)$ between the sets where $f <$ and $\leq y$ shows that these are the infima and suprema respectively of f on the S .

The sum of bounded place functions is defined [CL §95] as the sup of the sum sequence built from sequences of clopen step functions, with sups the respective summands, which in addition have their infs on every clopen convergent to the inf of the summand. The latter has the effect of making the lattice sup of the sequence its pointwise sup on X : since f is continuous, there exists for each x a clopen containing it on which $\inf f > f(x) - \varepsilon$, hence some $f_n(x) \geq \inf f_n \geq f(x) - \varepsilon$. Thus the place function sum is the same as the pointwise sum.

F. E. J. Linton has drawn my attention to an analogous discussion in [S §43, 44] (where references to other treatments can also be found). The focus of interest there is measurable functions modulo sets of measure zero: §43 relates these to Boolean σ -homomorphisms from the Borel subsets in the reals to the algebra of measurable modulo null sets and §44 relates them to the continuous functions on the latter's Stone space. Combining these would yield a correspondence between Boolean σ -homomorphisms and continuous functions, which could connect to the above correspondence between place and continuous functions—at least for the case of σ -rings—once it was shown that place functions extend uniquely to σ -homomorphisms. Unfortunately the treatment in [S §44] is inadequate: the first two equations are incorrect in that the function

h_0 is not applied to arguments in its domain and no proof is given that the resulting functions are continuous nor that they exhaust all continuous functions on the Stone space.

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