DISTRIBUTIONS AND THE LIE ALGEBRAS THEIR BASES CAN GENERATE

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ABSTRACT. The problem is to determine when a smooth, k -dimensional distribution D^k defined on an n -manifold M^n , locally admits a vector field basis which generates a nilpotent, solvable or even finite-dimensional Lie algebra. We show that for every $2 \leq k \leq n-1$ there exists a (nonregular at $p \in M^n$) distribution D^k on M^n which does not locally (near p) admit a vector field basis generating a solvable Lie algebra. From classical results on the equivalence problem, it is shown that for $1 \leq k \leq 4$ and D^k regular at $p \in M^4$, D^k admits a local vector field basis generating a nilpotent Lie algebra.

INTRODUCTION

Let $x \rightarrow D^k(x)$ be a real analytic, k -distribution defined on a nb d, $U(p)$, of a point p on the real analytic, n -dimensional manifold M^n . For $\{X^1, \dots, X^k\}$ a local vector field basis of D^k let $L = L(X^1, \dots, X^k)$ denote the Lie algebra these vector fields generate. The problem considered is to determine when a distribution D^k admits a basis generating a nilpotent, solvable, or finite-dimensional Lie algebra (hereafter termed a nilpotent, solvable, or finite basis for D^k). The main result, Theorem 3, leads to the fact that for every $2 \leq k \leq n-1$ there exists a k -distribution D^k on M^n which does not admit (locally) a solvable basis.

Let D^k, E^k be k -distributions defined, respectively, on n -manifolds M, N . Classically, $D^k(p)$ is *equivalent* to $E^k(q)$ if there is a local diffeomorphism $\phi: M \rightarrow N$, defined on a nb d $U(p) \subset M$, such that $\phi(p) = q$ and $\phi_*(x)D^k(x) = E^k(\phi(x))$ for $x \in U(p)$. We shall say $D^k(p)$ is *self-equivalent* to $D^k(q)$ if there exists a diffeomorphism ϕ , defined on a nb d $U(p)$, with $\phi(p) = q$ and $\phi_*(x)D^k(x) = D^k(\phi(x))$ for $x \in U(p)$. In [1], it was shown that if D^k admits a local nilpotent basis near p , then every nb d of p contains a point $q \neq p$ such that $D^k(p)$ and $D^k(q)$ are self-equivalent. In this

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case certain diffeomorphism invariants of distributions led to obstructions to the existence of local nilpotent bases.

In the solvable case, self-equivalence no longer need hold. We next describe, briefly, the procedure used to produce obstructions to the existence of a solvable basis.

For a Lie algebra L of vector fields on an n -manifold M^n with tangent space at x denoted TM_x^n , let $L(p) = \{X(p) \in TM_p^n : X \in L\}$ and L_p denote the isotropy algebra at p , i.e. $L_p = \{X \in L : X(p) = 0\}$. Then L is locally transitive at p if $\dim L(p) = n$. For D^k a local k -distribution on M^n its j th derived distribution is denoted $D^{k,j}$ and defined as follows. Choose any local basis $\{X^1, \dots, X^k\}$ for D^k . Then $D^{k,j}(x)$ is the span of all products of i -tuples of elements from $\{X^1, \dots, X^k\}$, evaluated at x , with $i \leq j+1$. This is a basis free notion. Note that $D^{k,0} = D^k$ while for $j \geq 1$, $D^{k,j}$ may be a singular distribution, i.e. $\dim D^{k,j}(x)$ need not be constant in a nbd of p . The integers $\dim D^{k,j}(x)$ are, however, diffeomorphism invariants, specifically if $D^k(p)$ is equivalent to $E^k(q)$, then $\dim D^{k,j}(p) = \dim E^{k,j}(q)$ for all $j = 1, 2, \dots$.

In the case of a locally transitive, solvable, algebra L , we first show (Theorem 2) that every nbd of p contains a point $q \neq p$ such that the isotropy subalgebras L_p, L_q are isomorphic and this isomorphism extends to an automorphism $\bar{l}$ of L . By a theorem of Nagano–Sussmann, [2, 3], here stated as Theorem 1, this implies that $D^k(p)$ and $(\bar{l}D^k)(q) = \text{span}\{(\bar{l}X^1)(q), \dots, (\bar{l}X^k)(q)\}$ are equivalent. Theorem 3 then shows that for q sufficiently near p the dimensions of the j th derived distributions of D^k and $\bar{l}D^k$ are the same at q for all $j = 1, 2, \dots$. Thus if D^k has a local solvable basis every nbd of p contains a point $q \neq p$ such that $\dim D^{k,j}(p) = \dim D^{k,j}(q)$, $i = 1, 2, \dots$. Example 1 constructs a D^k such that $\dim D^{k,2}(x) > \dim D^{k,2}(p)$ for all $x \neq p$ in some nbd of p . Such a D^k cannot admit a solvable basis. (One should note, however, that this behavior does not provide an obstruction to the existence of a basis generating a finite-dimensional algebra; see [4, Example 3]. This implies that the problem of finite-dimensional bases for D^k is distinct from the equivalence problem.) The third section of this paper shows the extent to which classical results on the equivalence problem can be utilized in the Lie algebra basis problem.

The motivation for this problem came from the study of control systems, on $\mathbf{R}^n$, of the form

$$(0.1) \quad \dot{x} = \sum_{i=1}^k u_i X^i(x), \quad x(0) = p$$

with the controls u_i taking values in $\mathbf{R}^1$. This system depends only on the distribution $D^k(x) = \text{span}\{X^1(x), \dots, X^k(x)\}$ and not on the specific basis

chosen for D^k . Indeed, the use of control "feedback" $u_i(x) = \sum_{j=1}^k a_{ij}(x)v_j$, $i = 1, \dots, k$, with $A = (a_{ij})$ a nonsingular, matrix-valued, function leads to the equivalent system $\dot{x} = \sum_{j=1}^k v_j Y^j(x)$ where $Y^j(x) = \sum_{i=1}^k a_{ij}(x)X^i(x)$, i.e. this is merely a basis change for D^k . If D^k admits a nilpotent (or solvable) basis the analysis of (1) is greatly simplified. Indeed, see [5], [6, §3], in this case the solutions for specific control choices, can be written as the compositions of flows corresponding to the vector fields in a properly chosen, ordered basis, for the algebra $L(X^1, \dots, X^k)$.

Systems of the form (0.1) arise naturally in the control of manipulator linkages. A less standard origin is in singular differential equations, on $\mathbf{R}^n$, of the form

$$(0.2) \quad B(x)\dot{x} = f(x), \quad x(0) = p$$

where B is an $n \times n$ matrix valued function with $\text{rank } B(x) = n - k$ for x in a nbd of p . If $D^k(x)$ denotes the null space of $B(x)$, the general solution of (0.2) is the sum of a specific solution of the nonhomogeneous problem with any solution of $\dot{x} \in D^k(x)$, this latter equation being of the form (0.1).

A final application is to the area of partial differential operators of the form $Tw = \sum_{i=1}^k (X^i)^2 w$ where $w : \mathbf{R}^n \rightarrow \mathbf{R}^1$. The operator T is hypoelliptic (at p) if $\dim L(X^1, \dots, X^k)(p) = n$. The fundamental singularity of the parametric for the problem $Tw = f$ depends on $D^k(x) = \text{span}\{X^1(x), \dots, X^k(x)\}$ and not on the specific basis chosen. The study of such a parametric involves an analysis of the shape of "geodesic balls" for a system such as (0.1) with control bounds $|u_i| \leq 1$. In control language, this is the shape of the attainable set at time $t = 1$. A detailed analysis can be found in [7].

Throughout this paper we work in the real analytic category. For X a vector field on M the notation $(\exp sX)(p)$ will denote the solution, at time s , of $\dot{x} = X(x)$, $x(0) = p$.

§1. THE THEOREM OF NAGANO-SUSSMANN

The papers [2, 3] study the equivalence question for Lie algebras of vector fields. Specifically, given transitive Lie algebras L, L' on manifolds M, M' , respectively, and a Lie algebra isomorphism $\bar{l} : L \rightarrow L'$, when is $\bar{l}$ induced by a diffeomorphism? The local version of the main theorem in [3] may be stated as follows.

Theorem (Sussman [3]). *Let L, L' be Lie algebras of real analytic vector fields on the real analytic, n -dimensional manifolds M, M' , respectively, and $p \in M, q \in M'$ such that $\dim L(p) = \dim L'(q) = n$. If there is a Lie algebra isomorphism $l : L_p \rightarrow L'_q$ which extends to an isomorphism $\bar{l} : L \rightarrow L'$, then there exist neighborhoods, $U(p)$ and $U'(q)$ and an analytic diffeomorphism $\phi : U(p) \rightarrow U'(q)$ with $\phi(p) = q$ and $\phi_*(x)X(x) = (\bar{l}X)(\phi(x))$ for all $X \in L, x \in U(p)$.*

By choosing $L = L'$ and $M = M'$, this theorem has immediate applications to the equivalence and self-equivalence problem for a distribution D^k . For later reference, we state this as

Corollary 1.1. *Let D^k be a real analytic, k -distribution having local basis $\{X^1, \dots, X^k\}$ defined in a nbd $U(p)$ with $L = L(X^1, \dots, X^k)$ such that $\dim L(p) = n$. If q , sufficiently near p , is such that L_p is isomorphic to L_q and the isomorphism l extends to an automorphism $\bar{l}$ of L , then $D^k(p)$ is equivalent to $(\bar{l}D^k)(q) = \text{span}\{(\bar{l}X^1)(q), \dots, (\bar{l}X^k)(q)\}$. If, furthermore,*

$$(1.1) \quad (\bar{l}X^i)(y) \in D^k(y), \quad i = 1, \dots, k; \quad y \in U(q)$$

then $D^k(p)$ is self-equivalent to $D^k(q)$.

Remark. 1. If $L(X^1, \dots, X^k)$ is nilpotent, it was shown in [1] that every nbd of p contains a point $q \neq p$ such that $L_p = L_q$. In this case one may choose $\bar{l} = \text{id}$ in Corollary 1.1 which then yields $D^k(p)$ self-equivalent to $D^k(q)$.

§2. DISTRIBUTIONS WITH SOLVABLE BASES

In this section we will prove, via a series of short propositions, the following.

Theorem 2. *Let $L = L(X^1, \dots, X^k)$ be a solvable Lie algebra of vector fields on $M = M^n$ with $\dim L(p) = n$. Then in every nbd of p there is a point $q \neq p$ such that the isotropy subalgebras L_p, L_q are isomorphic. Furthermore, if $l : L_p \rightarrow L_q$ is the Lie algebra isomorphism, l extends to a Lie algebra automorphism $\bar{l}$ of L .*

For $L = L(X^1, \dots, X^k)$ solvable, and say of dimension m over the reals, one can find (see [8, Corollary 3.7.5]) an ordered basis $\{Y^1, \dots, Y^m\}$ such that for each $1 \leq s \leq m$, $L(Y^1, \dots, Y^s)$ is an ideal, denoted G_s , in L . Specifically, there exist real-valued functions $\lambda_j : L \rightarrow \mathbb{R}$ such that

$$[X, Y^s] = \lambda_s(X)Y^s + \sum_{j=1}^{s-1} \lambda_j(X)Y^j, \quad s = 1, \dots, m, \quad X \in L.$$

Choose such a basis and let r denote the smallest integer such that $Y^r(p) \neq 0$. Such an r exists since $\dim L(p) = n$. We then have

Proposition 1.

$$(2.1) \quad [X, Y^r] = \lambda_r(X)Y^r + \sum_{j=1}^{r-1} \lambda_j(X)Y^j, \quad X \in L.$$

$$(2.2) \quad [X, Y^r](p) = \lambda_r(X)Y^r(p), \quad \text{i.e.} \quad \sum_{j=1}^{r-1} \lambda_j(X)Y^j \in G_{r-1} \subset L_p.$$

$$(2.3) \quad \lambda_r([X, Z]) = 0 \quad \text{for all } X, Z \in L.$$

Proof. Statements (2.1) and (2.2) are immediate from the choice of the ordered basis and integer r . For (2.3) we use the Jacobi identity, i.e.

$$\begin{aligned}
 [[X, Z], Y^r] &= [X, [Z, Y^r]] - [Z, [X, Y^r]] \\
 &= \left[X, \sum_{j=1}^r \lambda_j(Z) Y^j \right] - \left[Z, \sum_{j=1}^r \lambda_j(X) Y^j \right] \\
 &= \sum_{j=1}^r \lambda_j(Z) \sum_{i=1}^j \lambda_i(X) Y^i \\
 &\quad - \sum_{j=1}^r \lambda_j(X) \sum_{i=1}^j \lambda_i(Z) Y^i \\
 &= \sum_{i=1}^r \left(\sum_{j=i}^r \lambda_i(X) \lambda_j(Z) - \lambda_j(X) \lambda_i(Z) \right) Y^i.
 \end{aligned}$$

From this, one sees that the coefficient of Y^r , which is $\lambda_r([X, Z])$, is zero. $\square$

Proposition 2. $X \in L_p$ if and only if

$$(2.4) \quad X((\exp s Y^r)(p)) = s \lambda_r(X) Y^r((\exp s Y)(p)), \quad s \text{ real}.$$

(In words, $X \in L_p$ if and only if along the trajectory of Y^r thru p , X is a specific scalar multiple of Y^r .)

Proof. (a) If (2.4) holds, then for $s = 0$ we see $X \in L_p$.

(b) Suppose $X \in L_p$, i.e. $X(p) = 0$. Then

$$\begin{aligned}
 (i) \quad (\exp -s Y^r)_* X((\exp s Y^r)(p)) &= \sum_{\nu=0}^{\infty} ((-s)^\nu / \nu!) (ad^\nu Y^r, X)(p) \\
 &= X(p) + s[X, Y^r](p) = s \lambda_r(X) Y^r(p),
 \end{aligned}$$

where use was made of (2.2) in the last equality. But $(\exp -s Y^r)_*$ is a vector space isomorphism and $(\exp -s Y^r)_* Y^r((\exp s Y^r)(p)) = Y^r(p)$ hence it follows from (i) that $X((\exp s Y^r)(p)) = s \lambda_r(X) Y^r((\exp s Y^r)(p))$. $\square$

Next, let $|s| > 0$ be small and $q = (\exp s Y^r)(p)$. Define a map

$$(2.5) \quad l: L_p \rightarrow L, \quad l(X) = X - s \lambda_r(X) Y.$$

Proposition 3. The map l is linear, $l: L_p \rightarrow L_q$ and is a Lie algebra isomorphism.

Proof. The linearity of l follows immediately from its definition. Also, for $X \in L_p$, $l(X)(q) = X(q) - s \lambda_r(X) Y^r(q)$ which is zero by (2.4). Thus $l: L_p \rightarrow L_q$.

Next, we show l is one-one. Inded, for $X, Z \in L_p$ if $l(X) = l(Z)$, then $l(X)(p) = l(Z)(p) \Rightarrow \lambda_r(X) = \lambda_r(Z) \Rightarrow X = Z$.

Finally, to see that l is a Lie algebra isomorphism we show it preserves Lie products, i.e. that $l([X, Z]) = [l(X), l(Z)]$. To save repeating this argument for the extension $\bar{l}$ of l , one may note that no use will be made of the fact that here $X, Z \in L_p$. First, $l([X, Z]) = [X, Z]$ since $\lambda_r([X, Z]) = 0$ by (2.3). Next, we compute

$$\begin{aligned} [l(X), l(Z)] &= [X - s\lambda_r(X)Y^r, Z - s\lambda_r(Z)Y^r] \\ &= [X, Z] - s\lambda_r(Z)[X, Y^r] + s\lambda_r(X)[Z, Y^r] \\ &= [X, Z] - s\lambda_r(X)\lambda_r(Z)Y^r + s\lambda_r(X)\lambda_r(Z)Y^r = [X, Z], \end{aligned}$$

as required. $\square$

We now extend l to a mapping $\bar{l}: L \rightarrow L$ in the obvious way, i.e. define

$$(2.6) \quad \bar{l}: L \rightarrow L, \quad \bar{l}(X) = X - s\lambda_r(X)Y^r.$$

Proposition 4. *The mapping $\bar{l}$ is an extension of l and a Lie algebra automorphism of L .*

Proof. From the definition, $\bar{l}$ agrees with l in L_p . To see $\bar{l}$ is one-one, let $X, Z \in L$. Then $\bar{l}(X) = \bar{l}(Z) \Rightarrow X - s\lambda_r(X)Y^r = Z - s\lambda_r(Z)Y^r \Rightarrow (X - Z) = s(\lambda_r(X) - \lambda_r(Z))Y^r \Rightarrow (X(q) - Z(q)) = s(\lambda_r(X) - \lambda_r(Z))Y^r(q) \Rightarrow (X - Z) \in L_p$ (by Proposition 2) $\Rightarrow X(p) = Z(p) \Rightarrow \lambda_r(X) = \lambda_r(Z) \Rightarrow X = Z$.

The proof that $\bar{l}$ preserves Lie products is, as previously mentioned, the same as that given for l in Proposition 3. $\square$

Propositions 1–4 complete the proof of Theorem 2. Indeed, q can be chosen as any point on the (nonconstant) trajectory of Y^r through p .

An immediate consequence of Corollary 1.1 and Theorem 2 is that if D^k is a k -distribution on $M = M^n$ with local basis $\{X^1, \dots, X^k\}$ such that $L = L(X^1, \dots, X^k)$ is solvable and $\dim L(p) = n$, then every nbd of p contains a point $q \neq p$ such that $D^k(p)$ is equivalent to $\bar{l}D^k(q)$ where $\bar{l}D^k(x) = \text{span}\{(\bar{l}X^1)(x), \dots, (\bar{l}X^k)(x)\}$. Furthermore, we may note from the definition, (2.6), of $\bar{l}$ that if $Y^r \in D^k$, then $X^i \in D^k \Rightarrow \bar{l}X^i \in D^k$ and hence $\bar{l}D^k = D^k$ in which case we can conclude $D^k(p)$ is self-equivalent to $D^k(q)$. In general one cannot expect $Y^r \in D^k$, however, as we next show, for q sufficiently near p , $\bar{l}$ is near the identity hence $D^k(q)$ and $\bar{l}D^k(q)$ have certain relevant invariants in common, specifically the dimensions of the derived distributions.

Theorem 3. *Let D^k be a k -distribution on M^n with local basis $\{X^1, \dots, X^k\}$ such that $L = L(X^1, \dots, X^k)$ is solvable and $\dim L(p) = n$. Then every nbd of p contains a point $q \neq p$ such that $\dim D^{k,j}(p) = \dim D^{k,j}(q)$ for $j = 1, 2, \dots$.*

Proof. Let $\bar{l}$ be the Lie algebra automorphism as given by (2.6) and $q \neq p$ a point such that $D^k(p)$, $(\bar{l}D^k)(q)$ are equivalent.

By Theorem 1, $\exists$ a local diffeomorphism ϕ defined on a $nb\delta$. $U(p)$ such that $\phi(p) = q$ and $\phi_*(x)D^k(x) = \bar{L}D^k(\phi(x))$, $x \in U(p)$. Since $\phi_*(x)[X^i, X^j] \cdot (x) = [\bar{L}X^i, \bar{L}X^j](\phi(x))$ it clearly follows that $\dim D^{k,j}(p) = \dim \bar{L}D^{k,j}(q)$, $j = 1, \dots$, i.e. these dimensions are diffeomorphism invariants. What remains to be shown is that $\dim \bar{L}D^{k,j}(q) = \dim D^{k,j}(q)$ for q sufficiently near p .

Now $(\bar{L}X^i)(q) = X^i(q) - s\lambda_r(X^i)Y^r(q)$ and $(\bar{L}[X^i, X^j])(q) = [X^i, X^j](q)$, which follows since $\lambda_r([X^i, X^j]) = 0$ by (2.2). A well-known result of Dynkin (see [8; Ex. 44, p. 146]) shows when computing a generating set for $D^{k,j}$ it suffices to consider products of s -tuples of $X^1, \dots, X^k$ of the special form $[X^{i_1}, [X^{i_2}, [\dots [X^{i_{s-1}}, X^{i_s}] \dots]]]$. Thus $D^{k,j}(q) = \text{span}\{X^1(q), \dots, X^k(q), [X^{i_1}, [\dots [X^{i_{s-1}}, X^{i_s}] \dots]](q); s \leq j\}$ while the j th derived distribution of $\bar{L}D^k$, denoted $(\bar{L}D^k)^j$, satisfies

$$(\bar{L}D^k)^j(q) = \text{span}\{X^1(q) + s\lambda_r(X^1)Y^r(q), \dots, X^k(q) + s\lambda_r(X^k)Y^r(q), [X^{i_1}[\dots [X^{i_{s-1}}, X^{i_s}] \dots]](q); s \leq j\}.$$

From this, one sees for $|s| > 0$ but sufficiently small (i.e. for $q = (\exp sY^r) \cdot (p)$ sufficiently near p) $\dim D^{k,j}(q) = \dim \bar{L}D^{k,j}(q) = \dim D^{k,j}(p)$, $j = 1, \dots$. $\square$

Corollary 1. If $D^k(x) = D^{k,0}(x) = \text{span}\{X^1(x), \dots, X^k(x)\}$ is a local k -distribution on M^n with $\dim L(X^1, \dots, X^k)(p) = n$ and such that $\dim D^{k,1}(p) = k$ while $\dim D^{k,1}(x) > k$ if $x \neq p$, then D^k does not admit a solvable basis in any $nb\delta$ of p .

In [1, Example 1.1], for every $2 \leq k \leq n-1$ such a D^k was constructed hence one has

Corollary 2. For every $2 \leq k \leq n-1$ there exists a k -distribution D^k , defined on $\mathbb{R}^n$ which does not admit a solvable basis in any $nb\delta$ of zero.

For completeness, we illustrate this via the construction of a D^2 on $\mathbb{R}^3$ which admits no solvable basis in any $nb\delta$ of zero.

Example 1. Choose $X^1(x) = \partial/\partial x_1$, $X^2(x) = \partial/\partial x_2 - x_1((x_1^2/3) + x_2^2 + x_3^2)\partial/\partial x_3$ and $D^2(x) = \text{span}\{X^1(x), X^2(x)\}$. Then $[X^1, X^2](x) = (x_1^2 + x_2^2 + x_3^2)\partial/\partial x_3$, $(ad^2[X^1, X^2], X^1)(0) = 2\partial/\partial x_3$ hence $\dim L(X^1, X^2)(0) = 3$ while

$$\dim D^{2,1}(x) = \begin{cases} 2 & \text{if } x = 0 \\ 3 & \text{if } x \neq 0. \end{cases}$$

By Corollary 1, D^2 does not admit a solvable basis in any $nb\delta$ of zero.

§3. THE ROLE OF THE EQUIVALENCE PROBLEM IN THE BASIS PROBLEM

The equivalence problem for (locally defined) k -distributions on M^n may be viewed as their classification up to local diffeomorphisms. This problem was studied in the dual setting by Cartan [9], Goursat [10], Darboux, Von

Webber, etc. Given $D^k(x) = \text{span}\{X^1(x), \dots, X^k(x)\}$ defined in a *nb*d U of $p \in M^n$ let $\omega_1, \dots, \omega_{n-k}$ be linearly independent one-forms on M^n such that $\langle \omega_i(x), X^j(x) \rangle = 0$, $i = 1, \dots, n-k$, $j = 1, \dots, k$, for $x \in U$. Then the module I_{n-k} generated by $\omega_1, \dots, \omega_{n-k}$ over the ring of smooth, real valued, functions on M^n is called the *dual pfaffian system* to D^k . For example, if $k = n-1$, I_1 is generated by a single one form ω . The rank of ω at x is one-half the rank of the $(n-1) \times (n-1)$ skew symmetric matrix $S(x) = (\langle \omega(x), [X^i, X^j](x) \rangle)$. If ω has constant rank in a *nb*d of p , Dabroux's theorem gives a canonical form for ω relative to a specific choice of local coordinates. This, in turn, allows one to make an obvious choice of a dual vector field basis for D^{n-1} such that the elements of this basis generate a nilpotent Lie algebra. (See [1].) Thus although the Lie algebra structure of a vector field basis is independent of the local coordinates used to represent the basis elements, and although the equivalence classification does not take into account any specific bases, the classical results can give positive information concerning the existence of bases generating nice Lie algebras.

The classification of low dimensional pfaffian systems in the classical work of Cartan, Darboux, etc. was based on invariants, e.g. the rank $\omega(x)$ in the Darboux theorem, which were required to be locally constant. These invariants can be interpreted in the dual, or distribution, setting; our next task.

For any k -distribution D^k on M^n define its *Cauchy subdistribution* $C(D^k) = \{X \in D^k : [X, D^k] \subset D^k\}$. The Jacobi identity readily shows $C(D^k)$ is involutive. Again, the $D^{k,j}$ denote derived distributions. The invariants used by Cartan, Dabroux, etc. to classify pfaffian systems can be defined in terms of $\dim D^{k,j}(x)$, $\dim C(D^{k,j})(x)$ and hence to use the classical results, these dimensions should be locally constant. With this in mind we define D^k on M^n to be *regular at* $p \in M^n$ if for each $j \geq 0$, $\dim D^{k,j}(x)$, $\dim C(D^{k,j})(x)$ are constant in a *nb*d of p . One should remark that this is a slightly stronger notion of regular than that used by Hermann in [13]. Indeed, in [13], only $\dim D^{k,j}(x)$ locally constant was required while the invariants $C(D^{k,j})(x)$ were given an interesting characterization via integrability tensors.

For illustration, if D^{n-1} on $\mathbf{R}^n$ is regular at zero in the above sense and ω is a nonzero, dual, one form then (see [11, Theorem 1]) we have $2(\text{rank } \omega(x)) = (n-1) - \dim C(D^{n-1})(x)$. Thus D^{n-1} regular implies $\text{rank } \omega(x)$ is locally constant and the Darboux theorem applies. It is worth noting that the dimensions of Cauchy subdistributions are computable via linear algebra, see [11].

Example 2. ($\dim D^{k,j}(x)$ locally constant but $\dim C(D^{k,j})(x)$ not locally constant.)

Let $h(x) = -x_3x_1^2 - x_3x_2^2 - (x_3^2/2) - x_3x_4^2 - x_3x_5^2$. We consider $D^4(x) = \text{span}\{X^1(x), \dots, X^4(x)\}$ on $\mathbf{R}^5$ with $X^1 = \partial/\partial x_1$, $X^2 = \partial/\partial x_2 - x_1\partial/\partial x_5$, $X^3 = \partial/\partial x_3$, $X^4 = \partial/\partial x_4 + h(x)\partial/\partial x_5$. Since $[X^1, X^2] = \partial/\partial x_5$, $\dim D^{4,j}(x) = 5$ for $j \geq 1$ and x in a *nb*d of $0 \in \mathbf{R}^5$. A dual, nonzero, one form is

$\omega(x) = x_1 dx_2 - h(x) dx_4 + dx_5$ and a calculation shows

$$\text{rank } S(x) = \text{rank}(\langle \omega(x), [X^i, X^j](x) \rangle)_{1 \leq i, j \leq 4} = \begin{cases} 2 & \text{if } x = 0 \\ 4 & \text{if } x \neq 0. \end{cases}$$

Thus

$$\dim C(D^4)(x) = \begin{cases} 2 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

Here, D^4 is not regular.

A first example of the use of the classical equivalence results in the nilpotent basis problem can be stated as

Proposition 5. ([1]). *Every D^{n-1} on M^n which is regular at p admits a local, nilpotent basis which generates an algebra of dimension n .*

As a next example, we have

Proposition 6. *Every D^k on M^4 , $1 \leq k \leq 4$, which is regular at p admits a local, nilpotent, basis.*

Proof. For $k = 1$ or 4 , D^k is involutive hence has an Abelian basis; for $k = 3$ we may use Proposition 5 hence only the case $k = 2$ remains.

From [10, §75], the dual pfaffian system I_2 to D^2 may be taken to have generators ω_1, ω_2 which in preferred local coordinates take one of the following forms:

(A) $\omega_1 = -y_3 dy_1 + dy_2$, $\omega_2 = -y_4 dy_1 + dy_3$,

(B) $\omega_1 = dy_1$, $\omega_2 = -y_4 dy_2 + dy_3$,

(C) $\omega_1 = dy_1$, $\omega_2 = dy_2$.

For each case we choose a specific basis for the dual distribution D^2 , i.e. respectively:

(A') $D_1^2(y) = \text{span}\{X^1(y) = \partial/\partial y_4, X^2(y) = \partial/\partial y_1 + y_3\partial/\partial y_2 + y_4\partial/\partial y_3\}$,

(B') $D_2^2(y) = \text{span}\{X^1(y) = \partial/\partial y_4, X^2(y) = \partial/\partial y_2 + y_4\partial/\partial y_3\}$,

(C') $D_3^2(y) = \text{span}\{X^1(y) = \partial/\partial y_4, X^2(y) = \partial/\partial y_3\}$.

One may note that in case (C') we have D_3^2 involutive hence the Frobenius theorem gives the Abelian basis. In case (B'), $[X^1, X^2] = \partial/\partial x_3$ and all other brackets vanish. Here $\dim D_2^{2,1}(x) = 3$ and $D_2^{2,1}$ is involutive. Thus D_2^2 can be considered as a two distribution on the three-dimensional leaves of $D_2^{2,1}$ and the Darboux theorem can be used. Finally, in case (A'), $\dim D_1^{2,1}(x) = 3$, $\dim D_1^{2,2}(x) = 4$ which is sometimes called the Goursat-Von Weber case (see [12, Theorem 4.4]). In every case, the bases chosen generate a nilpotent Lie algebra. $\square$

Notice that the counterexamples to the existence of a nilpotent basis in [1] and to a solvable basis in §2, here, depended on D^k being not regular.

The case of D^2 on M^5 . The equivalence problem for two- and three-dimensional pfaffian systems in five variables was extensively studied by Cartan [9]; the results are also given by Goursat [10]. Dual to a three-dimensional pfaffian system on M^5 is a two-dimensional distribution which we will next analyze in detail. Similar results could be obtained for D^3 on M^5 . We will use invariants of the distribution D^k ; Cartan and Goursat use invariants of the dual pfaffian system I_{n-k} , mainly the class of I_{n-k} which is $\dim C(D^k)$. This invariant may also be defined via short exact sequences associated with I_{n-k} , see [14].

Let D^2 be a two distribution defined in a *nbd* of $0 \in \mathbf{R}^5$; be regular at zero and have dual pfaffian system denoted I_3 , with class $I_3 = 5$ (i.e., $\dim C(D^2)(x) = 0$ in a *nbd* of zero which is the generic case). Then if $D^2(x) = \text{span}\{X^1(x), X^2(x)\}$ we know $[X^1, X^2] \notin D^2$ i.e., $\dim D^{2,1}(x) = 3$. Dual to $D^{2,1}$ is the first derived pfaffian system which will necessarily be generated by two independent one forms. Following the notation of [10], we denote this as I'_2 . The classification (canonical forms) of I_3 , and hence D^2 , depend on the class of I'_2 , i.e. $\dim C(D^{2,1})$. The possibilities are (see also [10, p. 317]):

- (E) $\dim C(D^{2,1}) = 0$, i.e. $(ad^2 X^1, X^2) \notin D^{2,1}$, $(ad^2 X^2, X^1) \notin D^{2,1}$.
- (F) $\dim C(D^{2,1}) = 1$; one of $(ad^2 X^1, X^2)$, $(ad^2 X^2, X^1)$ belongs to $D^{2,1}$.
- (G) $\dim C(D^{2,1}) = 3$; $X^1, X^2, [X^1, X^2]$ are involutive.

Case (G). (See [10, p. 318].) Here one can choose a basis for I_3 of the form $\omega_1 = dy_1$, $\omega_2 = dy_2$, $\omega_3 = -y_5 dy_3 + dy_4$ hence in properly chosen local coordinates D^2 has a basis $\{X^1(y) = \partial/\partial y_5, X^2(y) = \partial/\partial y_3 + y_5 \partial/\partial y_4\}$ and this basis generates a nilpotent Lie algebra.

Case (F). (See [10, p. 318].) Here I_3 has either a basis $\omega_1 = dy_1$, $\omega_2 = y_4 dy_2 - dy_3$, $\omega_3 = y_5 dy_2 - dy_4$ and D^2 has $\{X^1 = \partial/\partial y_5, X^2 = \partial/\partial y_2 + y_4 \partial/\partial y_3 + y_5 \partial/\partial y_4\}$ as a nilpotent basis or I_3 has basis $\omega_1 = -y_3 dy_1 + dy_2$, $\omega_2 = -y_4 dy_1 + dy_2$, $\omega_3 = -y_5 dy_1 + dy_4$ and D^2 has $\{X^1 = \partial/\partial y_5, X^2 = \partial/\partial y_1 + y_3 \partial/\partial y_2 + y_4 \partial/\partial y_3 + y_5 \partial/\partial y_4\}$ as a nilpotent basis.

Case (E). (See [10, p. 320].) In this case I_3 was called *normal*; it is the generic case and I_3 has a basis of the form $\omega_1 = -y_3 dy_1 + dy_2$, $\omega_2 = -f dy_1 - y_5 dy_3 + dy_4$, $\omega_3 = (\partial f/\partial y_5) dy_1 + dy_3$ where $f = f(y_1, \dots, y_5)$ may be arbitrary except for $\partial^2 f(0)/\partial y_5^2 \neq 0$. Here D^2 has, as a canonical basis:

$$(3.1) \quad \left\{ X^1 = \frac{\partial}{\partial y_5}, X^2 = \frac{\partial}{\partial y_1} + y_3 \frac{\partial}{\partial y_2} - \left(\frac{\partial f}{\partial y_5} \right) \frac{\partial}{\partial y_3} + \left(f - y_5 \frac{\partial f}{\partial y_5} \right) \frac{\partial}{\partial y_4} \right\}.$$

It is easy to see that one can choose functions f , with $\partial^2 f(0)/\partial y_5^2 \neq 0$, in (3.1) such that the Lie algebra generated by X^1, X^2 is infinite dimensional. This does not mean, for such an f , that a change of basis could not yield a new basis generating a finite Lie algebra, i.e. the Lie algebra basis problem becomes distinct from the equivalence problem.

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