ON THE INTERSECTION OF A CLASS OF MAXIMAL SUBGROUPS OF A FINITE GROUP

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(Communicated by Warren J. Wong)

ABSTRACT. Let G be a finite group and π a set of primes. We consider the family of subgroups of G : $\mathcal{F} = \{M: M < \cdot G, [G: M]_{\pi} = 1, [G: M] \text{ is composite}\}$ and denote $S_{\pi}(G) = \bigcap \{M: M \in \mathcal{F}\}$ if $\mathcal{F}$ is non-empty, otherwise $S_{\pi}(G) = G$. The purpose of this note is to prove

Theorem. Let G be a π -solvable group. Then $S_{\pi}(G)$ has the following properties: (1) $S_{\pi}(G)/O_{\pi}(G)$ is supersolvable. (2) $S_{\pi}(S_{\pi}(G)) = S_{\pi}(G)$. (3) $G/O_{\pi}(G)$ is supersolvable if and only if $S_{\pi}(G) = G$.

There has been much interest in the past in considering various generalisations of the Frattini subgroup of a finite group and to investigate the influence of such a subgroup on the structure of the group (see, Deskins [1], Gaschütz [2], Rose [3], and [4]). In [4], P. Bhattacharya and N. P. Mukherjee introduce a subgroup $S_{\pi}(G)$ and exhibit its relationship with the given group G . The objective of this paper is to investigate the subgroup $S_{\pi}(G)$ further. We obtain the following result.

Theorem. Let G be a finite π -solvable group, then $S_{\pi}(G)/O_{\pi}(G)$ is a supersolvable group.

Let π be any set of primes and π' the complementary set of primes. Let G be a finite group. Then we denote $M < \cdot G$ to indicate that M is a maximal subgroup of G . Also, $[G: M]_{\pi}$ denotes the π -part of $[G: M]$. Now consider the following family of subgroups:

$$\mathcal{F} = \{M: M < \cdot G, [G: M]_{\pi} = 1, [G: M] \text{ is composite}\}.$$

Definition. $S_{\pi}(G) = \bigcap \{M: M \in \mathcal{F}\}$ if $\mathcal{F}$ is non-empty, otherwise $S_{\pi}(G) = G$. Clearly, $S_{\pi}(G) \trianglelefteq G$, $\Phi(G) \leq S_{\pi}(G)$ and $O_{\pi}(G) \leq S_{\pi}(G)$, where $\Phi(G)$ is the Frattini subgroup of the group G and $O_{\pi}(G)$ is the maximal normal π -subgroup of G .

All groups in this paper are finite.

Received by the editors September 22, 1988.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 20D20.

Key words and phrases. Finite group, maximal subgroup.

Lemma 1. Let H be a solvable normal subgroup of a group G , $1 \neq N \leq H$, $N \trianglelefteq G$, and let H/N be a supersolvable group. Then $G = N \cdot N_G(V)$, where V is a supersolvable projector of H and is not normal in G .

Proof. By [5, Theorem 5, 1, 7] a supersolvable projector V of H exists, and $H = NV$. Consequently $H = NV^g$ for every $g \in G$ since $H \trianglelefteq G$. It is proved easily that V^g is a supersolvable projector of H also. Thus, there is $h \in H$ such that $V^g = V^h$, and $g \in N_G(V) \cdot H$. The result now follows.

Lemma 2. If G is a π -solvable group, then $S_\pi(G)$ is a solvable group.

Proof. We assume that $S_\pi(G) \neq 1$. Let N be a minimal normal subgroup of G with $N \leq S_\pi(G)$. Then by induction hypothesis $S_\pi(G)/N$ is solvable because $S_\pi(G/N) = S_\pi(G)/N$ ([4, Lemma 1]). Since G is π -solvable, we have that N is either a π' -group or a p -group for some prime $p \in \pi$. If N is a p -group, then $S_\pi(G)$ is solvable. If N is a π' -group, then N is contained in every π' -Hall subgroup of G and $N \leq S_{\pi'}(G)$. Consequently, $N \leq S_\pi(G) \cap S_{\pi'}(G)$. By [4, Lemma 2] and [4, Theorem 3] $S_\pi(G) \cap S_{\pi'}(G)$ is supersolvable and N is solvable. Thus, $S_\pi(G)$ is a solvable group. The Lemma is proved.

Proof of Theorem. Since $O_\pi(G/O_\pi(G)) = 1$ and $S_\pi(G/O_\pi(G)) = S_\pi(G)/O_\pi(G)$, we may assume that $O_\pi(G) = 1$ and $S_\pi(G) \neq 1$.

Case 1. G has a minimal normal subgroup $N \leq S_\pi(G)$ such that $O_\pi(G/N) = 1$. By induction hypothesis $S_\pi(G)/N$ is supersolvable. Let V be a supersolvable projector of $S_\pi(G)$. Then $S_\pi(G) = NV$.

If $V \trianglelefteq G$, then $V \cap N \not\leq G$. Since N is a minimal normal subgroup $N \cap V = 1$ or N . Consequently, $S_\pi(G) = N \times V$ or V . Notice that V is a supersolvable projector of $S_\pi(G)$. Thus, $S_\pi(G) = V$ is supersolvable.

If $V \not\trianglelefteq G$, then by Lemma 1 there is a maximal subgroup M of G such that $N_G(V) \leq M$ and $G = NM$. Notice that N is an abelian p -group for some prime $p \in \pi'$; thus $M \cap N = 1$ and $[G: M]_\pi = 1$. If $[G: M]$ is composite, then $N \leq S_\pi(G) \leq M$ and $G = M$, contradiction. Thus, $[G: M]$ is a prime and N is a cyclic group of prime order. Consequently $S_\pi(G)$ is supersolvable.

Case 2. For every minimal normal subgroup $N(\leq S_\pi(G))$ of G , $O_\pi(G/N) \neq 1$.

Let $N \leq S_\pi(G)$ be a minimal normal subgroup of G , $|N| = p^\alpha$, $p \in \pi'$, and let $T/N \leq S_\pi(G)/N$ be a minimal normal π -subgroup of G/N , $|T/N| = q^\beta$ for some prime $q \in \pi$. If $Q \in \text{Syl}_q T$, then by a Frattini argument $G = N \cdot N_G(Q)$. Since $Q \not\trianglelefteq G$, there is a maximal subgroup M of G such that $N_G(Q) \leq M$ and $G = NM$. It is obvious that $N \not\leq \Phi(G)$. Since N is a minimal normal subgroup $M \cap N = 1$, $[G: M]_\pi = 1$. Consequently, $[G: M]$ is a prime and N is a cyclic group of prime order. Thus, we have proved the following:

(1) N is a cyclic group of prime order for every minimal normal subgroup $N(\leq S_\pi(G))$ of G .

(2) $\Phi(G) = 1$.

(3) The fitting subgroup $F(S_\pi(G))$ of $S_\pi(G)$ is the direct product of minimal normal subgroups of G which are contained in $S_\pi(G)$.

In fact, let H be maximal among all subgroups of $F(S_\pi(G))$ which can be expressed as the direct product of minimal normal subgroups of G . Note then that $H \trianglelefteq G$. We will first show that there is a subgroup S of G such that $G = HS$ and $H \cap S = 1$.

Let S be a minimal element of set $\{T: HT = G, T \leq G\}$. Then, $S \cap H \trianglelefteq G$ ($S \cap H \trianglelefteq S$ because $H \trianglelefteq G$; $S \cap H \trianglelefteq H$ because H is abelian; hence $S \cap H \trianglelefteq SH = G$). If $S \cap H \neq 1$, then since $\Phi(G) = 1$, there is a maximal subgroup M of G such that $G = M(S \cap H)$. We now see that $S = S \cap G = (S \cap M)(S \cap H)$ and, since M does not contain $S \cap H$, that $S \cap M < S$. However, $G = SH = (S \cap M)(S \cap H)H = (S \cap M)H$. This contradicts the choice of S and we conclude that $S \cap H = 1$.

We next consider the subgroup $S \cap F(S_\pi(G))$. It is normal in both S and in $F(S_\pi(G))$ (the latter because $\Phi(G) = 1$ implies $\Phi(F(S_\pi(G))) = 1$, but $[F(S_\pi(G))]' \leq \Phi(F(S_\pi(G)))$ since $F(S_\pi(G))$ is nilpotent, hence $F(S_\pi(G))$ is abelian). Thus, it is normal in G . If $S \cap F(S_\pi(G)) \neq 1$, then there is a minimal normal subgroup N of G with $N \leq S \cap F(S_\pi(G))$. As $S \cap H = 1$, we conclude that $H < N \times H$. This contradicts the choice of H and therefore $S \cap F(S_\pi(G)) = 1$. It now follows that $H = F(S_\pi(G))$ and the result is proved.

(4) $S_\pi(G)$ is supersolvable.

Let M be a maximal subgroup of $S_\pi(G)$. If $M \not\geq F(S_\pi(G))$, then M does not contain some minimal normal subgroup N of G which is contained in $S_\pi(G)$ by (3). Hence $S_\pi(G) = MN$ and $M \cap N = 1$. By (1), N is a cyclic group of prime order and therefore $[S_\pi(G): M]$ is a prime. Since $[S_\pi(G): M] = [F(S_\pi(G)): F(S_\pi(G)) \cap M]$, it follows that $M \cap F(S_\pi(G))$ is a maximal subgroup of $F(S_\pi(G))$. By [5, Theorem 1, 3, 3] $S_\pi(G)$ is a supersolvable group.

Corollary 1. *Let G be a π -solvable group. Then a π' -Hall subgroup H of $S_\pi(G)$ is supersolvable.*

Corollary 2. *Let G be a π -solvable group. Then $S_\pi(S_\pi(G)) = S_\pi(G)$.*

Proof. Since $S_\pi(G)/O_\pi(G)$ is supersolvable and $O_\pi(G)$ is contained in every π -Hall subgroup of $S_\pi(G)$, every maximal subgroup of $S_\pi(G)$ which contains a π -Hall subgroup of $S_\pi(G)$ is of prime index in $S_\pi(G)$, or there is no maximal subgroup which contains a π -Hall subgroup of $S_\pi(G)$ in $S_\pi(G)$. Hence $S_\pi(S_\pi(G)) = S_\pi(G)$.

Corollary 3. *Let G be a π -solvable group. Then $G/O_\pi(G)$ is supersolvable if and only if $G/S_\pi(G)$ is supersolvable.*

Proof. Since $O_\pi(G) \leq S_\pi(G)$, the necessity is obvious. Let $G/S_\pi(G)$ be supersolvable. Then $S_\pi(G/S_\pi(G)) = G/S_\pi(G)$, but by [4, Lemma 1] $S_\pi(G/S_\pi(G)) = S_\pi(G)/S_\pi(G) = 1$, therefore, $S_\pi(G) = G$ and $G/O_\pi(G) = S_\pi(G)/O_\pi(G)$ is supersolvable by the theorem.

It is obvious that Corollary 3 is equivalent to

Corollary 4. *If G is π -solvable, then $G/O_\pi(G)$ is supersolvable if and only if $S_\pi(G) = G$.*

REFERENCES

1. W. E. Deskins, *On maximal subgroups*, First Sympos. Pure Math., Amer. Math. Soc. Providence, RI, 1959.
2. W. Gaschütz, *Über die Untergruppe endlicher Gruppen*, Math. Z. **58** (1953), 260–270.
3. J. Rose, *The influence on a group of its abnormal structure*, J. Math. Lond. Soc. **40** (1965), 348–361.
4. P. Bhattacharya and N. P. Mukherjee, *On the intersection of a class of maximal subgroups of a finite group II*, J. Pure and Applied Algebra (North-Holland) **42** (1986), 117–124.
5. M. Weistein, *Between nilpotent and solvable*, Polygonal Publishing House, 1982.

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