THE CAUCHY FUNCTIONAL EQUATIONS IN DISTRIBUTIONS

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ABSTRACT. The Pompeiu functional equation is defined by Neagu for Schwartz distributions. His method is extended to the four Cauchy functional equations by means of two new operators Q^* and R^* on $\mathcal{D}'(I)$. The Cauchy equations in distributions reduce to the classical equations when the solutions are regular distributions, i.e. locally integrable functions.

1. INTRODUCTION

Recently [1], the Pompeiu equation has been defined for distributions in $\mathcal{D}'(\Omega)$ where $\Omega = (-1, \infty) \subset \mathbf{R}$. When the distribution is a locally integrable function $f(x)$, the equation in distribution reduces to the classical Pompeiu equation

$$f(x + y + xy) = f(x) + f(y) + f(x)f(y).$$

Fenyő [2], in finding the most general solution to the Hosszú equation

$$f(x + y - xy) + f(xy) = f(x) + f(y),$$

has defined and solved it in distributional form. In this paper, we define the Cauchy equations in distributions by means of two new operators Q^* and R^* on $\mathcal{D}'(I)$. When the solutions are locally integrable functions, the equations reduce to the classical Cauchy functional equations [3].

- (1) $f(x + y) = f(x) + f(y)$
- (2) $f(x + y) = f(x)f(y)$
- (3) $f(xy) = f(x) + f(y)$
- (4) $f(xy) = f(x)f(y).$

2. PRELIMINARIES

Let $I = (0, \infty) \subset \mathbf{R}$ and $I^2 = I \times I \subset \mathbf{R}^2$. We denote by $\mathcal{D}(I)$ and $\mathcal{D}(I^2)$ the spaces of infinitely differentiable functions on I and I^2 with compact supports, respectively. Similarly, $\mathcal{E}(I)$ and $\mathcal{E}(I^2)$ are spaces of infinitely

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differentiable functions on I and I^2 . The duals of these spaces are denoted by a prime, e.g. $\mathcal{D}'(I)$. Note that $\mathcal{D}(I) \subset \mathcal{E}(I) \subset \mathcal{E}'(I) \subset \mathcal{D}'(I)$ (see [4]). We denote by $L_{\text{loc}}(I)$ and $L_{\text{loc}}(I^2)$ the spaces of locally integrable functions on I and I^2 , respectively.

D is the differentiation operator in $\mathcal{D}'(I)$, whereas D_1 and D_2 are the partial differentiation operators in $\mathcal{D}'(I^2)$ with respect to the first and second variable from I^2 , respectively.

3. SOME OPERATORS ON $\mathcal{D}'$

Let E_1 and E_2 be integration operators from $\mathcal{D}(I^2)$ into $\mathcal{D}(I)$ given, respectively, by

$$(5) \quad E_1[\phi](x) = \int_I \phi(x, y) dy$$

and

$$(6) \quad E_2[\phi](y) = \int_I \phi(x, y) dx$$

for any $\phi \in \mathcal{D}(I^2)$. It is easy to see that both of (5) and (6) are linear operators and we denote this by membership in $\mathcal{L}[\mathcal{D}(I^2); \mathcal{D}(I)]$. The adjoints of E_1 and E_2 are the operators E_1^* and E_2^* from $\mathcal{D}'(I)$ into $\mathcal{D}'(I^2)$ defined by

$$(7) \quad \langle E_1^*[T], \phi \rangle = \langle T, E_1\phi \rangle = \left\langle T_x, \int_I \phi(x, y) dy \right\rangle$$

$$(8) \quad \langle E_2^*[T], \phi \rangle = \langle T, E_2\phi \rangle = \left\langle T_y, \int_I \phi(x, y) dx \right\rangle$$

for $T \in \mathcal{D}'(I)$ and $\phi \in \mathcal{D}(I^2)$. We note that $E_i^* \in \mathcal{L}[\mathcal{D}'(I); \mathcal{D}'(I^2)]$, $i = 1, 2$. The following is proved in an analogous way to [1] after replacing the domains Ω and Ω^2 by I and I^2 , respectively,

Proposition 1.

(a) If $f \in L_{\text{loc}}(I)$, then $g = E_1^*[f] \in L_{\text{loc}}(I^2)$ and $h = E_2^*[f] \in L_{\text{loc}}(I^2)$, where $g(x, y) = f(x)$ and $h(x, y) = f(y)$.

(b) If $\alpha \in \mathcal{E}(I)$, then $\beta = E_1^*[\alpha] \in \mathcal{E}(I^2)$ and $\gamma = E_2^*[\alpha] \in \mathcal{E}(I^2)$.

(c) If $\alpha \in \mathcal{E}(I)$ and $T \in \mathcal{D}'(I)$, then

$$(9) \quad E_i^*[\alpha T] = E_i^*[\alpha]E_i^*[T], \quad i = 1, 2.$$

(d) If $T \in \mathcal{D}'(I)$, then

$$(10) \quad D_1 E_1^*[T] = E_1^*[DT]; \quad D_1 E_2^*[T] = 0$$

$$(11) \quad D_2 E_1^*[T] = 0; \quad D_2 E_2^*[T] = E_2^*[DT].$$

Let P be the direct product operator from $\mathcal{D}'(I) \times \mathcal{D}'(I)$ into $\mathcal{D}'(I^2)$ given by

$$(12) \quad \langle P[S; T], \phi \rangle = \langle S_x, \langle T_y, \phi(x, y) \rangle \rangle = \langle T_y, \langle S_x, \phi(x, y) \rangle \rangle$$

for any $S, T \in \mathcal{D}'(I)$ and $\phi \in \mathcal{D}(I^2)$. It is known that P is a linear operator and that

$$(13) \quad D_1 P[S; T] = P[DS; T]; \quad D_2 P[S; T] = P[S; DT]$$

(see [4]). The following result from [1] is immediate, *mutatis mutandis*. (Note that equations (14) as stated in [1] are incorrect.)

Proposition 2.

(a) If $T \in \mathcal{D}'(I)$, then we have

$$(14) \quad E_1^*[T] = P[T; 1]; \quad E_2^*[T] = P[1; T].$$

(b) If $\alpha, \beta \in \mathcal{E}(I)$ and $S, T, U \in \mathcal{D}'(I)$, then we have

$$(15) \quad E_1^*[\alpha]P[S; U] + E_1^*[\beta]P[T; U] = P[\alpha S + \beta T; U]$$

$$(16) \quad E_2^*[\alpha]P[S; T] + E_2^*[\beta]P[S; U] = P[S; \alpha T + \beta U].$$

(c) If $S, T, U, V \in \mathcal{D}'(I)$ are nonzero distributions and satisfy the equation

$$(17) \quad P[S; T] = P[U; V],$$

then $S = c_1 U$ and $T = c_2 V$ (where c_1 and c_2 are real constants) and conversely, (17) is satisfied for all real c_1 and c_2 such that $c_1 = 1/c_2$.

4. THE OPERATORS Q^* AND R^*

Let Q and R be operators from $\mathcal{D}(I^2)$ into $\mathcal{D}(I)$ given by

$$(18) \quad Q[\phi](u) = \int_I \phi(u-v, v) dv = \int_I \phi(v, u-v) dv$$

$$(19) \quad R[\phi](u) = \int_I \frac{\phi(u/v, v)}{v} dv = \int_I \frac{\phi(v, u/v)}{v} dv.$$

These are well-defined since $\phi(x, y)$ has compact support. We note that $Q, R \in \mathcal{L}[\mathcal{D}(I^2); \mathcal{D}(I)]$. The adjoints of these operators are $Q^*, R^* \in \mathcal{L}[\mathcal{D}'(I); \mathcal{D}'(I^2)]$ defined by

$$(20) \quad \langle Q^*[T], \phi \rangle = \langle T, Q[\phi] \rangle = \langle T_u, Q[\phi](u) \rangle$$

$$(21) \quad \langle R^*[T], \phi \rangle = \langle T, R[\phi] \rangle = \langle T_u, R[\phi](u) \rangle.$$

Proposition 3.

(a) If $f \in L_{\text{loc}}(I)$, then $Q^*[f] \in L_{\text{loc}}(I^2)$ where $Q^* f = f(x+y)$.

(b) If $f \in L_{\text{loc}}(I)$, then $R^*[f] \in L_{\text{loc}}(I^2)$ where $R^* f = f(xy)$.

Proof. (a) For any $\phi \in \mathcal{D}(I^2)$, we have

$$\begin{aligned} \langle Q^*[f], \phi \rangle &= \langle f, Q[\phi] \rangle = \left\langle f(u), \int_I \phi(u-v, v) dv \right\rangle \\ &= \int_{I^2} f(u) \phi(u-v, v) dv du \\ &= \langle f(x+y), \phi(x, y) \rangle. \end{aligned}$$

(b) For any $\phi \in \mathcal{D}(I^2)$, we have

$$\begin{aligned}\langle R^*[f], \phi \rangle &= \left\langle f(u), \int_I \frac{\phi(u/v, v)}{v} dv \right\rangle \\ &= \int_{I^2} \frac{f(u)\phi(u/v, v)}{v} dv du \\ &= \langle f(xy), \phi(x, y) \rangle.\end{aligned}$$

Proposition 4. If $\alpha(x) \in \mathcal{E}(I)$, then both $Q^*(\alpha)$ and $R^*(\alpha)$ belong to $\mathcal{E}(I^2)$.

Proof. Analogous to Proposition 3.

Proposition 5. If $\alpha \in \mathcal{E}(I)$ and $T \in \mathcal{D}'(I)$, then

$$(22) \quad Q^*[\alpha T] = Q^*[\alpha]Q^*[T]$$

$$(23) \quad R^*[\alpha T] = R^*[\alpha]R^*[T].$$

Proof. For any $\phi \in \mathcal{D}(I^2)$, we have

$$\begin{aligned}\langle R^*[\alpha T], \phi \rangle &= \langle \alpha T, R[\phi] \rangle = \langle T, \alpha R[\phi] \rangle \\ &= \langle T, R[\phi R^*[\alpha]] \rangle = \langle R^*[T], \phi R^*[\alpha] \rangle \\ &= \langle R^*[\alpha]R^*[T], \phi \rangle.\end{aligned}$$

The first statement is proved in the same way.

Proposition 6. If $T \in \mathcal{D}'(I)$, then

$$(24) \quad D_1(Q^*[T]) = Q^*(DT) = D_2(Q^*[T]).$$

$$(25) \quad D_1(R^*[T]) = E_2^*(\Omega)R^*(DT)$$

$$(26) \quad D_2(R^*[T]) = E_1^*(\Omega)R^*(DT)$$

where $\Omega = x \in \mathcal{E}(I)$.

Proof. Let $\phi(x, y) \in \mathcal{D}(I^2)$ and $\psi(x, y) = -\partial\phi/\partial x$. Then

$$\begin{aligned}\langle D_1(Q^*[T]), \phi(x, y) \rangle &= \langle Q^*[T], \psi(x, y) \rangle = \langle T, Q\psi \rangle \\ &= \left\langle T_u, \int_I \psi(u-v, v) dv \right\rangle = \left\langle T_u, \int_I -\frac{\partial\phi}{\partial u}(u-v, v) dv \right\rangle \\ &= \left\langle T_u, -\frac{\partial}{\partial u} \int_I \phi(u-v, v) dv \right\rangle \\ &= \langle DT, Q(\phi) \rangle = \langle Q^*[DT], \phi \rangle.\end{aligned}$$

Since $\int_I \phi(u-v, v) dv = \int_I \phi(v, u-v) dv$, the second part of (24) is proved analogously. To prove (25), we have

$$\begin{aligned} \langle D_1(R^*[T]), \phi(x, y) \rangle &= \langle R^*[T], \psi(x, y) \rangle = \langle T, R\psi \rangle \\ &= \left\langle T_u, \int_I \frac{\psi(u/v, v)}{v} dv \right\rangle \\ &= \left\langle T_u, \int_I \frac{-v}{v} \frac{\partial \phi}{\partial u} \left(\frac{u}{v}, v \right) dv \right\rangle \\ &= \left\langle T_u, \frac{-\partial}{\partial u} \int_I \phi \left(\frac{u}{v}, v \right) dv \right\rangle = \langle DT_u, R[\phi v] \rangle \\ &= \langle R^*[DT], \phi \cdot E_2^*(\Omega) \rangle \quad \text{where } \Omega = x \in \varepsilon(I) \\ &= \langle E_2^*(\Omega) R^*[DT], \phi \rangle. \end{aligned}$$

Equation (26) is proved analogously.

5. THE CAUCHY EQUATIONS IN DISTRIBUTIONS

Using the operators Q^* and R^* , we shall call the following equations the Cauchy equations in distribution $T \in D'(I)$:

$$(27) \quad Q^*[T] = E_1^*[T] + E_2^*[T]$$

$$(28) \quad Q^*[T] = P[T; T]$$

$$(29) \quad R^*[T] = E_1^*[T] + E_2^*[T]$$

$$(30) \quad R^*[T] = P[T; T].$$

Proposition 7. If $T = f \in L_{\text{loc}}(I)$, equations (27) to (30) reduce to the Cauchy functional equations (1) to (4), respectively.

Proof. The statement follows from Proposition 1(a), Proposition 3, and the fact that for $\phi \in \mathcal{D}(I^2)$,

$$\begin{aligned} \langle P[f; f], \phi \rangle &= \langle f(x), \langle f(y), \phi(x, y) \rangle \rangle \\ &= \int_{I^2} f(x)f(y)\phi(x, y) dx dy \\ &= \langle f(x)f(y), \phi(x, y) \rangle. \end{aligned}$$

Theorem 1. If $T \in \mathcal{D}'(I)$ satisfies (27), then $T \in \mathcal{E}(I)$ where $T = c\Omega$, $c \in \mathbf{R}$.

Proof. From Propositions 6, 1(d) and 2(a), we have

$$Q^*[DT] = D_1 Q^*[T] = D_1 E_1^*[T] + D_1 E_2^*[T] = P[DT; 1]$$

$$Q^*[DT] = D_2 Q^*[T] = D_2 E_1^*[T] + D^2 E_2^*[T] = P[1; DT].$$

Hence $P[1; DT] = P[DT; 1]$ and by Proposition 2(c), $DT = c$ or $T = c\Omega$, $c \in \mathbf{R}$.

Corollary 1. If $f \in L_{\text{loc}}(I)$ satisfies (27), then $f(x) \in \mathcal{E}(I)$ where $f(x) = cx$, $c \in \mathbf{R}$.

This is the classical solution of the Cauchy functional equation (1).

Theorem 2. If $T \in \mathcal{D}'(I)$ satisfies (28), then $T \in \mathcal{E}(I)$ where $T = e^{c\Omega}$, $c \in \mathbf{R}$ or $T \equiv 0$.

Proof. From Propositions 6, 1(d) and equation (13), we have

$$Q^*[DT] = D_1 Q^*[T] = D_1 P[T; T] = P[DT; T]$$

$$Q^*[DT] = D_2 Q^*[T] = D_2 P[T; T] = P[T; DT].$$

Hence $P[DT; T] = P[T, DT]$ and by Proposition 2(c), $DT = cT$ or $T = c_1 e^{c\Omega}$, $c_1, c \in \mathbf{R}$. On substituting back into (28), we obtain $c_1 = 1$ or 0 . Thus $T = e^{c\Omega}$ or $T \equiv 0$.

Corollary 2. If $f \in L_{\text{loc}}(I)$ satisfies (28), then $f(x) \in \mathcal{E}(I)$ where $f(x) = e^{cx}$ or 0 , $c \in \mathbf{R}$.

Theorem 3. If $T \in D'(I)$ satisfies (29), then $T \in \mathcal{E}(I)$ where $T = c \ln \Omega$, $c \in \mathbf{R}$.

Proof. From Propositions 6, 1(d) and 2(a), we have

$$E_2^*(\Omega)R^*(DT) = D_1 R^*[T] = D_1 E_1^*[T] + D_1 E_2^*[T] = P[DT; 1]$$

$$E_1^*(\Omega)R^*(DT) = D_2 R^*[T] = D_2 E_1^*[T] + D_2 E_2^*[T] = P[1; DT].$$

Hence $E_1^*(\Omega)P[DT; 1] = E_2^*(\Omega)P[1; DT]$. From Proposition 2(b) and 2(c), we obtain $\Omega DT = c$, $c \in \mathbf{R}$. Solving the differential equation in T , we get $T = c \ln \Omega + a$ for some arbitrary constant a . Substituting the last expression into the equation, we see that $a = 0$ and the theorem is proved.

Corollary 3. If $f \in L_{\text{loc}}(I)$ satisfies (29), then $f(x) \in \mathcal{E}(I)$ where $f(x) = c \ln x$, $c \in \mathbf{R}$.

This is the well-known solution of the Cauchy equation (3). In a similar manner, the following result can easily be shown.

Theorem 4. If $T \in \mathcal{D}'(I)$ satisfies (30), then $T \in \mathcal{E}(I)$ where $T = \Omega^c$, $c \in \mathbf{R}$ or $T \equiv 0$.

Corollary 4. If $f \in L_{\text{loc}}(I)$ satisfies (30), then $f(x) \in \mathcal{E}(I)$ where $f(x) = x^c$, $c \in \mathbf{R}$ or $f(x) \equiv 0$.

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