ON THE CLASSIFICATION OF HOMOGENEOUS MULTIPLIERS BOUNDED ON $H^1(\mathbf{R}^2)$

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ABSTRACT. Necessary and sufficient conditions for Calderon-Zygmund singular integral operators to be bounded operators on $H^1(\mathbf{R}^2)$ are investigated. Let m be a bounded measurable function on the circle, extended to $\mathbf{R}^2$ by homogeneity ($m(rx) = m(x)$). If the Calderon-Zygmund singular integral operator T_m , defined by $T_m f = \mathcal{F}^{-1}(m\mathcal{F}(f))$, is bounded on $H^1(\mathbf{R}^2)$, then it is proved that S^*m has bounded variation on the circle, where the Fourier transform of S on the circle is $\widehat{S}(n) = (-\operatorname{sgn}(n))^{n+1}$. This implies that m must have an absolutely convergent Fourier series on the circle, and other relations on the Fourier series of m . Partial converses are also given. The problems are formulated in terms of distributions on the circle and on $\mathbf{R}^2$.

1. INTRODUCTION

The principal subject of this paper is classification of Calderon-Zygmund singular integrals that are bounded operators on $H^1(\mathbf{R}^2)$. These operators have kernels which are homogeneous of degree -2 and multipliers which are homogeneous of degree 0 . We treat the problem in the general context of distributions. More precisely, suppose that m is a function or distribution on $\mathbf{R}^2$ which is homogeneous of degree 0 and let U be the distribution on $\mathbf{R}^2$ defined by $\mathcal{F}(U) = m$, where $\mathcal{F}$ denotes the Fourier transform on $\mathbf{R}^2$. We investigate necessary and sufficient conditions on m so that U extends to a bounded operator on $H^1(\mathbf{R}^2)$. We use the definition of $H^1(\mathbf{R}^2)$ in terms of atomic decompositions, as in Coifman and Weiss [2].

A first necessary condition is that m be a bounded measurable function, for if the multiplier operator $T_m(f) = \mathcal{F}^{-1}(m\mathcal{F}(f))$ is a bounded operator on H^1 , then by duality we have $T_m: BMO \rightarrow BMO$ continuously. By interpolation it follows that $T_m: L^2 \rightarrow L^2$ continuously, and hence that m must be a bounded measurable function. The problem is what additional conditions

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upon m force $T_m \in B(H^1)$, the bounded operators on $H^1(\mathbf{R}^2)$. We obtain necessary conditions in terms of the bounded variation of certain convolutions with m . We also prove that these necessary conditions are sufficient for T_m to map L^p boundedly to L^p for $p > 1$. In [12], Taibleson and Weiss have shown that if T_m is in $B(H^p)$ for $p \leq 1$ then m is continuous on $\mathbf{R}^2 \setminus \{0\}$. The replacement of L^1 by H^1 for this study is natural as it is well known that T_m is not bounded on L^1 if m is not constant. In fact, $T_m(L^1)$ is not contained in L^1 . See Stein [10, p. 42] for specifics.

In related work, Daly [4] provided a complete classification of those homogeneous multiplier operators that send atoms to molecules boundedly in $H^p(\mathbf{R}^n)$ for $0 < p \leq 1$. This classification, the L^p result indicated above (see Theorem 7) and conditions found by Taibleson and Weiss [12] are the most complete sufficiency results for $\mathbf{R}^2$.

The corresponding extension and classification problem for homogeneous multipliers for local fields is solved in [5].

In §2 we analyze homogeneous distributions and obtain a polar decomposition for them. The results will be given for $\mathbf{R}^n$, although in this paper we will only use the results for $n = 2$. The results in §2 are of independent interest, and will provide a basis for the corresponding H^1 problem in $\mathbf{R}^n$ for $n > 2$.

For notational purposes we will follow Hörmander [8] as much as possible. Both [8] and Donoghue [6] contain results on homogeneous distributions. For H^p theory we follow Coifman, Taibleson, and Weiss [2, and 12].

2. HOMOGENEOUS DISTRIBUTIONS

Three subsets of $\mathbf{R}^n$ play a role in our analysis: $\mathbf{R}^n$, $\mathbf{R}^n \setminus \{0\}$, and the unit sphere $\sum_{n-1}$. If X is one of the three, let $C_0^\infty(X)$ denote the infinitely differentiable functions on X having compact support. For a compact subset K of X and a nonnegative integer k , let

$$\|\varphi\|_{K,k} = \sum_{|\alpha| \leq k} \sup\{|D^\alpha \varphi(x)| : x \in K\}.$$

These seminorms define the topology of $C_0^\infty(X)$; see [8, p. 34]. The space $D'(X)$ of distributions on X is the dual of $C_0^\infty(X)$ with this topology. So-called test functions are elements of $C_0^\infty(X)$. In the case $X = \sum_{n-1}$, the space X is itself compact and so the only K needed in the above definition is $\sum_{n-1}$; and, $C_0^\infty(\sum_{n-1}) = C^\infty(\sum_{n-1})$.

The value of U in D' at φ is denoted $\langle U, \varphi \rangle$. A distribution U in $D'(\mathbf{R}^n)$ or $D'(\mathbf{R}^n \setminus \{0\})$ is said to be homogeneous of degree a , for $a \in \mathbf{R}$, if

$$\langle U, \varphi \rangle = t^{a+n} \langle U, \varphi_t \rangle$$

where $\varphi_t(x) = \varphi(tx)$ for $t > 0$.

If u is a distribution in $D'(\sum_{n-1})$, then u can be used to obtain a distribution U_a on $\mathbf{R}^n \setminus \{0\}$ that is homogeneous of degree a , in the following manner.

For $a \in \mathbf{R}$ and $\varphi \in C_0^\infty(\mathbf{R}^n \setminus \{0\})$, define U_a by

$$\langle U_a, \varphi \rangle = \int_0^\infty r^{n-1+a} \langle u, \varphi_r \rangle dr$$

where φ_r is the function defined on $\sum_{n-1}$ by $\varphi_r(x') = \varphi(rx')$. As φ has compact support, the integral above converges absolutely and defines a linear functional U_a which is in $D'(\mathbf{R}^n \setminus \{0\})$. The distribution u is said to be the kernel of the distribution U_a .

If $a = -(n+k)$ and k is not a nonnegative integer, then U_a has a unique extension to a distribution $\tilde{U}_a$ in $D'(\mathbf{R}^n)$. See Hörmander [8], Theorem 3.2.3. If k is a nonnegative integer, then U_a has an extension $\tilde{U}_a$ in $D'(\mathbf{R}^n)$ if and only if $\langle U_a, x^\alpha \psi \rangle = 0$ for radial $\psi \in C_0^\infty(\mathbf{R}^n \setminus \{0\})$ and all nonnegative integer multi-indices α such that $|\alpha| = k$. To obtain this condition and hence the extension to $\mathbf{R}^n$, it is easy to see that we must have $\langle u, x^\alpha \rangle = 0$ for $|\alpha| = k$. If $k = 0$, this condition reduces to $\langle u, 1 \rangle = 0$. The homogeneous extension $\tilde{U}_a$ in this case is only unique up to a linear combination of the derivatives of order k of the Dirac measure δ . This is because derivatives of order k of δ are homogeneous of order $-(n+k)$ and are supported at zero. See Theorem 3.2.4 of Hörmander [8]. For the purposes of this paper, we will be interested in the case $(n, k, a) = (2, 0, -2)$ and the extension where the contribution of δ is zero.

Suppose next that $U \in D'(\mathbf{R}^n)$ is homogeneous of degree a . From Donoghue [6, p. 154], U is a tempered distribution. That is, U has a unique continuous extension to the space S of rapidly decreasing functions on $\mathbf{R}^n$. We will show that there is a $u \in D'(\sum_{n-1})$ for which $U = U_a$. We need the following two lemmas.

Lemma 1. *If $U \in D'(\mathbf{R}^n)$ is homogeneous of degree $a \geq -n$, then U has finite order.*

Proof. If $\varphi \in C_0^\infty(\mathbf{R}^n)$ and has support in the unit ball B^1 , there exists a constant c and an integer N such that

$$(i) \quad |\langle U, \varphi \rangle| \leq c \cdot \|\varphi\|_{B^1, N}.$$

For any $\varphi \in C_0^\infty(\mathbf{R}^n)$, there exists $r \geq 1$ such that $\text{supp}(\varphi) \subset B^r$, the ball of radius r . Then $\varphi_{1/r}$ has support in B^1 . Thus

$$|\langle U, \varphi_{1/r} \rangle| \leq c \cdot \|\varphi_{1/r}\|_{B^1, N} \leq c \cdot \sum_{k=0}^N r^{-k} \sum_{|\alpha|=k} \|D^\alpha \varphi\|_\infty \leq c \cdot \|\varphi\|_{B^r, N}.$$

As U is homogeneous of degree a , $\langle U, \varphi_{1/r} \rangle = r^{a+n} \langle U, \varphi \rangle$. So we have

$$|\langle U, \varphi \rangle| \leq r^{-a-n} c \cdot \|\varphi\|_{B^r, N}.$$

Thus U has finite order if $a \geq -n$. (If N is the smallest integer for which (i) holds, then the order of U is N .) $\square$

Lemma 2. *If the distribution U is homogeneous of degree $a \geq -n$, then there exists a continuous function g and a multi-index α such that, for $\varphi \in C_0^\infty(\mathbf{R}^n)$,*

$$(ii) \quad \langle U, \varphi \rangle = \langle D^\alpha g, \varphi \rangle = \langle g, (-1)^{|\alpha|} D^\alpha \varphi \rangle.$$

Proof. From Donoghue [6, p. 106], a distribution of finite order N is an $(N+2)$ derivative of a continuous function. $\square$

We can now state and prove the main theorem on the correspondence between homogeneous distributions on $\mathbf{R}^n$ and $\mathbf{R}^n \setminus \{0\}$ and distributions on Σ_{n-1} .

Theorem 3. *Let $a \in \mathbf{R}$.*

(1) *If $u \in D'(\Sigma_{n-1})$ and $a \neq -(n+k)$ where k is a nonnegative integer, define U for $\varphi \in C_0^\infty(\mathbf{R}^n \setminus \{0\})$ by*

$$\langle U, \varphi \rangle = \int_0^\infty r^{n-1+a} \langle u, \varphi_r \rangle dr.$$

Then U is in $D'(\mathbf{R}^n \setminus \{0\})$ and has a unique extension to a distribution in $D'(\mathbf{R}^n)$ that is homogeneous of degree a .

(2) *If $U \in D'(\mathbf{R}^n)$ is homogeneous of degree a and a is not of the form $-(n+k)$ for a nonnegative integer k , then there exists $u \in D'(\Sigma_{n-1})$ such that*

$$\langle U, \varphi \rangle = \int_0^\infty r^{n-1+a} \langle u, \varphi_r \rangle dr.$$

for $\varphi \in C_0^\infty(\mathbf{R}^n \setminus \{0\})$.

(3) *If $a = -(n+k)$ and k is a nonnegative integer, then (1) holds if $\langle u, x^\alpha \rangle = 0$ for each nonnegative integer multi-index α with $|\alpha| = k$. In this case the extension to $D'(\mathbf{R}^n)$ is unique only up to addition of a linear combination of derivatives of order k of δ . If U in $D'(\mathbf{R}^n)$ is homogeneous of degree a as in (2), then u exists and $\langle u, x^\alpha \rangle = 0$ must hold for $|\alpha| = k$.*

Proof. We have proved (1). To prove (2), first suppose that $a > -n$. Then Lemma 2 applies and equality (ii) can be rewritten as

$$\begin{aligned} \langle U, \varphi \rangle &= \int_{\mathbf{R}^n} (-1)^{|\alpha|} g(x) D^\alpha(\varphi)(x) dx \\ &= (-1)^{|\alpha|} \int_0^\infty r^{n-1} \int_{\Sigma_{n-1}} g(rx') D^\alpha(\varphi)(rx') dx' dr \\ &= (-1)^{|\alpha|} \int_0^\infty r^{n-1+a} \int_{\Sigma_{n-1}} (r^{-a-|\alpha|} g(rx')) D^\alpha(\varphi_r)(x') dx' dr. \end{aligned}$$

Consider the function $h(rx') = r^{-(a+|\alpha|)} g(rx')$. The function h is continuous on $\mathbf{R}^n \setminus \{0\}$ and, as U is homogeneous of degree a , h is homogeneous of degree 0. Thus U can be written as

$$(iii) \quad \langle U, \varphi \rangle = \int_0^\infty r^{n-1+a} \langle u, \varphi_r \rangle dr$$

where u is defined by

$$(iv) \quad \langle u, \sigma \rangle = \int_{\sum_{n-1}} h(x') (-1)^{|\alpha|} D^\alpha(\sigma)(x') dx'$$

for $\sigma \in C^\infty(\sum_{n-1})$. The derivative $D^\alpha(\sigma)$ is defined by extending σ radially. Explicitly, choose a radial $\psi \in C^\infty(\mathbf{R}^n \setminus \{0\})$ that is identically equal to 1 on a neighborhood of $\sum_{n-1}$, and let $D^\alpha(\sigma)(x') = D^\alpha(\psi\sigma)(x')$. Then $u \in D'(\sum_{n-1})$ and U is the U_a defined by u . Note that for given $\varphi \in C_0^\infty(\mathbf{R}^n)$, the function $\langle u, \varphi_r \rangle$ has compact support on $\mathbf{R}^+$ and is a bounded function of r , so the function $\{r^{n-1+a} \langle u, \varphi_r \rangle\}$ is in $L^1(\mathbf{R}^1)$. Hence the integral in (iii) is finite.

If $a < -n$, then $|\cdot|^{-a-n}U$ is a distribution in $D'(\mathbf{R}^n \setminus 0)$ which is homogeneous of degree $-n$. Hence by Lemma 2 and the above argument there is a continuous function g on $\mathbf{R}^n$ and a multi-index α such that the function h defined by

$$h(x) = |x|^{-(n+|\alpha|)} g(x)$$

and the distribution u defined by

$$\langle u, \sigma \rangle = \int_{\sum_{n-1}} h(x') (-1)^{|\alpha|} D^\alpha(\sigma)(x') dx'$$

satisfy

$$\langle |\cdot|^{-a-n}U, \varphi \rangle = \int_0^\infty r^{-1} \langle u, \varphi_r \rangle dr$$

for $\varphi \in C_0^\infty(\mathbf{R}^n \setminus 0)$. The polar decomposition equality (iii) follows for U .

If $a = -(n+k)$ for a nonnegative integer k and if $u \in D'(\sum_{n-1})$, then the formula in (1) defines an element U of $D'(\mathbf{R}^n \setminus \{0\})$ which is homogeneous of degree a . The extension to $\mathbf{R}^n$ exists if $\langle u, x^\alpha \rangle = 0$ for all α satisfying $|\alpha| = k$. Conversely, if U is in $D'(\mathbf{R}^n \setminus \{0\})$ and is homogeneous of degree $a = -(n+k)$, then the definition of u in (iv) is the same as before and (iii) holds for $\varphi \in C_0^\infty(\mathbf{R}^n \setminus \{0\})$. If U extends to $D'(\mathbf{R}^n \setminus \{0\})$, then $\langle U, x^\alpha \psi \rangle = 0$ must hold for radial ψ . It follows easily that $\langle u, x^\alpha \rangle = 0$ must hold. Hence we have proved (3). $\square$

Further remarks. The formulas in Theorem 3 hold for $\varphi \in C_0^\infty(\mathbf{R}^n)$ if $a > -n$ but the extensions from $\mathbf{R}^n \setminus \{0\}$ to $\mathbf{R}^n$ are given by limits if $a \leq -n$. Since U is homogeneous it is also tempered and so in fact U extends continuously to S as well. It is easy to see that the formulas hold for those φ in S for which φ is zero on a neighborhood of 0 and for all φ in S if $a > -n$.

If U is homogeneous of degree a , then the Fourier transform $\mathcal{F}(U)$ is homogeneous of degree $n+a$; it too is tempered. In the case $a = -n$, $\mathcal{F}(U)$ is homogeneous of degree 0. The element m of $D'(\sum_{n-1})$ corresponding to $\mathcal{F}(U)$ as in Theorem 3, is called the multiplier of U . The distribution U defines in the usual way an operator T_m on $C_0^\infty(\mathbf{R}^n)$ by

$$T_m(\varphi)(x) = \langle U, \tau_x \varphi \rangle$$

where τ_x is translation by $-x$ and $\phi(x) = \phi(-x)$. Since U is tempered, this formula extends to $\phi \in S$. This paper concerns the extension of T_m from S as an operator on Hardy spaces.

3. MULTIPLIER OPERATORS

In the remainder of this paper we will be concerned only with distributions on $\mathbf{R}^2$ that are homogeneous of degree -2 . The distribution u in $D'(\Sigma_1)$ plays the role of the usual kernel in the construction of Calderon-Zygmund singular integrals. We let $\Sigma_1 = T$, the circle. We will always assume that U has no delta measure component (recall that δ is homogeneous of degree -2). The Fourier transform of U is homogeneous of degree 0 and so there is a unique distribution m on T satisfying

$$\langle \mathcal{F}(U), \phi \rangle = \int_0^\infty \langle m, \phi_r \rangle r dr.$$

In this case the relationship between u and m can be stated explicitly, in terms of the Fourier transform on the circle T , as

$$(1) \quad \hat{u}(n) = n(\text{isgn}(n))^{n+1} \cdot \hat{m}(n)/2\pi i$$

for $n \neq 0$, where $\text{sgn}(n)$ denotes the sign of n .

We give two proofs of (1). From Stein [10], for dimension 2, m can be computed from u on T by

$$m = -[(i\pi/2) \text{sgn}(\cos(\cdot)) + \log |\cos(\cdot)|] * u.$$

A straightforward computation gives, for $n \neq 0$,

$$-[(i\pi/2) \text{sgn}(\cos(\cdot)) + \log |\cos(\cdot)|]^\wedge(n) = 2\pi i(\text{isgn}(n))^{-(n+1)}/n.$$

Expression (1) follows immediately.

The formula (1) can also be derived by directly computing the Fourier transform of the distribution $e^{in\theta}/|\cdot|^2$ in $D'(\mathbf{R}^2 \setminus 0)$ using the methods of the proof of Theorem 5, *infra*. We give this derivation, which is heuristic in that existence of certain integrals needs to be established.

Let $\chi_n(\theta) = e^{in\theta}$, $n \neq 0$. Each $\chi_n \cdot |\cdot|^a$ for a real is locally integrable on $\mathbf{R}^n \setminus \{0\}$ and so defines a distribution. Let $\phi \in C_0^\infty(\mathbf{R}^2 \setminus 0)$. Using Theorem 3 and letting $y = se^{i\beta}$, we have

$$\begin{aligned} \langle \mathcal{F}(\chi_n |\cdot|^{-2}), \phi \rangle &= \langle \chi_n |\cdot|^{-2}, \mathcal{F}(\phi) \rangle \\ &= \int_0^\infty \int_0^{2\pi} \frac{1}{r} e^{in\theta} \int_{\mathbf{R}^2} \phi(se^{i\beta}) e^{-i\langle se^{i\beta}, re^{i\theta} \rangle} dy d\theta dr \\ &= \int_{\mathbf{R}^2} \phi(se^{i\beta}) \int_0^\infty \frac{1}{r} \int_0^{2\pi} e^{in\theta} e^{-i\langle se^{i\beta}, re^{i\theta} \rangle} d\theta dr dy. \end{aligned}$$

The inner-product in $\mathbf{R}^2$ can be rewritten using $\langle e^{i\beta}, e^{i\theta} \rangle = \cos(\theta - \beta)$. The θ -integral is a Bessel function, namely

$$\int_0^{2\pi} e^{in\theta} e^{-isr \cos(\theta - \beta)} d\theta = 2\pi e^{in\beta} (-i)^n J_n(rs)$$

where J_n is the n th-Bessel function. Using the multiplicative invariance of the measure $r^{-1} dr$ on $\mathbf{R}^+$, the equality $J_{-n}(r) = (-1)^n J_n(r)$ for $n > 0$, and integration formulas from Watson [13, p. 391, no. 385], the r -integral becomes

$$\int_0^\infty \frac{1}{r} J_n(rs) dr = (\operatorname{sgn}(n))^{n+1} / n .$$

Thus

$$\langle \mathcal{F}(\chi_n | \cdot|^{-2}, \varphi) \rangle = (2\pi i^{-n} (\operatorname{sgn}(n))^{n+1} / n) \langle \chi_n, \varphi \rangle .$$

We have shown that if we take $u = \chi_n$, then $m = c_n \chi_n$ where c_n is the constant on the right. The relationship (1) between the Fourier transforms of u and m on $\mathbf{T}$ follows for any m and u .

Since the degree of U is -2 , for the extension of U to $\mathbf{R}^2$ it is necessary that $\hat{u}(0) = 0$. Since we have assumed that U has no delta measure component, it is also true that $\hat{m}(0) = 0$.

Define the convolution operator S on $C^\infty(\mathbf{T})$ by

$$\hat{S}(n) = 2\pi(-i \operatorname{sgn}(n))^{n+1} .$$

The operator S will play a central role in the classification of those homogeneous multipliers that give rise to bounded operators on $H^1(\mathbf{R}^2)$. If we let τ_θ denote translation by θ , then a straightforward computation shows that S is invertible with inverse

$$S^{-1} = (1/4\pi^2) \tau_\pi S .$$

In fact, S can be written as

$$S = (1/2) \tau_{\pi/2} \{H(I + \tau_\pi) + i(I - \tau_\pi)\}$$

where H is the Hilbert transform on the circle, defined by $\hat{H}(n) = -i \operatorname{sgn}(n)$.

In terms of S , the relationship (1) between u and m is

$$\hat{u}(n) = (2\pi)^{-2} i n (-1)^n \cdot \hat{S}(n) \hat{m}(n) .$$

As we observed in the introduction, if T_m extends to a bounded operator on $L^2(\mathbf{R}^2)$, then m must be a bounded measurable function on $\mathbf{T}$. Thus $S * m$ is a function and we have

$$(2\pi)^2 i (-1)^{n+1} \hat{u}(n) / n = (S * m)^\wedge(n) .$$

Denoting the distributional derivative by D , it follows that we may write u in $D'(\mathbf{T})$ as

$$u = -D(S^{-1} * m) .$$

From Edwards [7, Theorem 12.5.16], u is a measure if and only if $\hat{u}(n)/n$ is the Fourier transform of a function of bounded variation. Hence we have the following lemma.

Lemma 4. *The function $S * m$ has bounded variation if and only if u is a measure.*

We will now prove the main result of the paper.

Theorem 5. Suppose m is a bounded measurable function on $\mathbf{R}^2$ which is homogeneous of degree 0. If the singular integral operator T_m is in $B(H^1)$, then the functions $S * m$, $S * (\sin(\cdot)m)$, and $S * (\cos(\cdot)m)$ have bounded variation on $\mathbf{T}$.

Proof. Assume T_m is in $B(H^1)$. Let η be a radial function in $C^\infty(\mathbf{R}^2)$ that is supported on $\{x : 1/4 \leq |x| \leq 4\}$ and is identically equal to 1 on $\{x : 1/2 \leq |x| \leq 2\}$. Then $M = \mathcal{F}^{-1}(\eta)$ is in $H^1(\mathbf{R}^2)$; see Taibleson and Weiss [12, p. 137]. Thus $T_m(M)$ is in $L^1(\mathbf{R}^2)$. For $a > 0$, we have

$$\begin{aligned} \|T_m(M)\|_1 &\geq \int_{\mathbf{R}^2} e^{-a|z|} |T_m(M)|(z) \, dz \\ (2) \quad &= \int_{\mathbf{R}^2} e^{-a|z|} |\mathcal{F}^{-1}(m\eta)(z)| \, dz \\ &= \int_{\mathbf{R}^2} e^{-a|z|} \left| \int_0^\infty \int_0^{2\pi} \eta(rx') m(rx') e^{ir\langle x', z \rangle} \, dx' r \, dr \right| \, dz. \end{aligned}$$

If m is expanded in a Fourier series, $\{A_n\}$ is any decomposition of $\mathbf{R}^2$, x' is replaced by $e^{i\theta}$, and z equals $se^{i\beta}$, then (2) becomes

$$\|T_m(M)\|_1 \geq \sum_n \int_{A_n} e^{-as} \left| \int_0^\infty \eta(r) \sum_k \hat{m}(k) \int_0^{2\pi} e^{ik\theta} e^{irs \cdot \cos(\theta - \beta)} \, d\theta r \, dr \right| \, dz.$$

Performing the θ -integration, we obtain

$$(3) \quad \|T_m(M)\|_1 \geq \sum_n \left| \int_{A_n} e^{-as} \int_0^\infty \eta(r) \sum_k \hat{m}(k) 2\pi (-i)^k e^{ik\beta} J_k(rs) r \, dr \, dz \right|$$

where J_k is the k th-Bessel function. If we let $\{A_n\}$ be the conical decomposition of $\mathbf{R}^2$ induced by a decomposition $0 = b_0 < b_1 < \cdots < b_N = 2\pi$ for the circle, (3) becomes

$$(4) \quad \|T_m(M)\|_1 \geq \sum_n \left| \int_0^\infty \eta(r) \sum_k \hat{m}(k) 2\pi (-i)^k \int_{b_{n-1}}^{b_n} e^{ik\beta} \, d\beta \int_0^\infty e^{-as} J_k(rs) s \, ds r \, dr \right|.$$

Using $J_k = (\operatorname{sgn}(k))^k J_{|k|}$ and an integration formula from Watson [13, p. 385], the s -integral can be explicitly evaluated as

$$(5) \quad \frac{(\operatorname{sgn}(k))^k (r/2)^k (1 + |k|)}{(a^2 + r^2)^{(2+|k|)/2}} \cdot {}_2F_1((|k|+2)/2, (|k|-1)/2; |k|+1; r^2/(a^2 + r^2))$$

where ${}_2F_1$ is a generalized hypergeometric function. Call the function displayed in (5) $C(k, a, r)$. For $a > 0$, $r\eta(r)C(k, a, r)$ is integrable and is dominated by $r\eta(r)C(k, 0, r)$. The function $C(k, 0, r)$ is given by

$$\begin{aligned} C(k, 0, r) &= \frac{(\operatorname{sgn}(k))^k (1 + |k|)}{2^k r^2} \cdot {}_2F_1((|k|+2)/2, (|k|-1)/2; |k|+1; 1) \\ &= (\operatorname{sgn}(k))^k |k| r^{-2} \end{aligned}$$

by using Gauss's Formula for ${}_2F_1(\alpha, \beta; \gamma; 1)$ (see Bateman Manuscript [1, p. 61]). Substituting this into (4) and performing the β -integration, we obtain the bound

$$\begin{aligned} \|T_m(M)\|_1 &\geq C_\eta \cdot \sum_n \left| \sum_k 2\pi(-i)^{k+1} \operatorname{sgn}(k)^{k+1} \hat{m}(k)(e^{-ikb_n} - e^{-ikb_{n-1}}) \right| \\ &= C_\eta \cdot \sum_n |S * m(e^{ib_n}) - S * m(e^{ib_{n-1}})|. \end{aligned}$$

As $\|T_m(M)\|_1$ is finite, $S * m$ is of bounded variation on $\mathbb{T}$.

To show that $S * (\sin(\cdot)m)$ and $S * (\cos(\cdot)m)$ must also be of bounded variation, we consider the two Riesz transforms R_1 and R_2 , defined by $\mathcal{F}(R_1)(re^{i\theta}) = i \cdot \sin(\theta)$ and $\mathcal{F}(R_2)(re^{i\theta}) = i \cdot \cos(\theta)$. These operators are bounded on H^1 ; and hence, $R_1 T_m$ and $R_2 T_m$ are bounded on H^1 if T_m is bounded. Applying the preceding argument to these operators, we see that, $S * (\sin(\cdot)m)$ and $S * (\cos(\cdot)m)$ are of bounded variation. $\square$

Theorem 6. *Suppose m is a bounded measurable function on $\mathbb{R}^2$ which is homogeneous of degree 0 and such that T_m is in $B(H^1)$. Then m has an absolutely convergent Fourier series as a function on $\mathbb{T}$.*

Proof. From Theorem 5, if $T_m \in B(H^1)$, then $S * m$, $S * (\sin(\cdot)m)$, and $S * (\cos(\cdot)m)$ are of bounded variation on $\mathbb{T}$. Thus $(-\operatorname{sgn}(n))^{n+1} \hat{m}(n)$, $(-\operatorname{sgn}(n))^{n+1}(\hat{m}(n+1) + \hat{m}(n-1))$, and $(-\operatorname{sgn}(n))^{n+1}(\hat{m}(n+1) - \hat{m}(n-1))$ are the Fourier series of functions of bounded variation; and consequently, $(-\operatorname{sgn}(n))^{n+1} \hat{m}(n+1)$ is the Fourier transform of a function of bounded variation. Taken together, all this implies that the Fourier transform

$$(H * S * m)^\wedge(n) = (-\operatorname{sgn}(n)) \cdot (-\operatorname{sgn}(n))^{n+1} \hat{m}(n)$$

is a finite sum of transforms of functions of bounded variation. From Zygmund [14, v. 1, p. 242], as both $S * m$ and $H * S * m$ are of bounded variation, $S * m$ has an absolutely convergent Fourier series. However, $|(S * m)^\wedge(n)| = 2\pi|\hat{m}(n)|$ and so m has an absolutely convergent Fourier series. $\square$

Theorem 7. *Suppose m is a bounded measurable function on $\mathbb{R}^2$ which is homogeneous of degree 0 and T_m is in $B(H^1)$. Then u is a measure on $\mathbb{T}$ and u is absolutely continuous with respect to Lebesgue measure.*

Proof. Theorem 5 and Lemma 4 imply that u is a measure. The kernel of $H * m$ is $H * u$, because both H and the correspondence between kernels and multipliers are linear. In the proof of Theorem 6 it was shown that $H * S * m$ has finite variation. Since $H * S = S * H$, it follows that $S * H * m$ has finite variation. Hence, by Lemma 4, $H * u$ is also a measure. By the F. Riesz and M. Riesz Theorem, u is absolutely continuous. See Zygmund [14, v. 1, p. 285], and also Edwards [7, p. 99]. $\square$

Theorem 6 can be phrased for kernels u . If u is a kernel in $D'(\mathbb{T})$ and $T_m \in B(H^1)$, then by Theorem 7 u is in $L^1(\mathbb{T})$. By Theorem 6, m has an

absolutely convergent Fourier series as a function on $\mathbb{T}$. However, $|\hat{m}(n)| = |\hat{u}(n)|/|n|$ if $n \neq 0$, and so $\sum_{n \neq 0} \frac{|\hat{u}(n)|}{|n|} < \infty$. We have proved:

Corollary 8. *If u is in $L^1(\mathbb{T})$ and the corresponding singular integral T_m is in $B(H^1)$, then $\sum_{n \neq 0} \frac{|\hat{u}(n)|}{|n|} < \infty$.*

We now provide a partial converse to Theorem 5.

Theorem 9. *Suppose m is a bounded measurable function on $\mathbb{R}^2$ which is homogeneous of degree 0. If $S * m$, $S * (\cos(\cdot)m)$, $S * (\sin(\cdot)m)$ are of bounded variation on $\mathbb{T}$, then T_m is in $B(L^p)$ for $1 < p < \infty$.*

Proof. Let $BV(\mathbb{T})$ be the functions of bounded variation and suppose $S * m$, $S * (\cos(\cdot)m)$, $S * (\sin(\cdot)m) \in BV(\mathbb{T})$. A periodic function f on $\mathbb{R}$ is even or odd as a function on the circle according as $f(t) = f(t+\pi)$ or $f(t) = -f(t+\pi)$. Note that $\cos(t)$ as a function on $\mathbb{T}$ is odd, not even. The even and odd parts of a function f are given by

$$f_e(\theta) = (f(\theta) + f(\theta + \pi))/2, \quad f_o(\theta) = (f(\theta) - f(\theta + \pi))/2.$$

Clearly $f \in BV(\mathbb{T})$ if and only if f_o and $f_e \in BV(\mathbb{T})$. As S is a multiplier operator, $(S * m)_o = S * (m_o)$ and $(S * m)_e = S * (m_e)$. For the odd function m_o , we have $m_o \in BV(\mathbb{T})$ as S reduces to a translation operator for odd functions. By Lemma 4, the corresponding kernel u_o is a measure. As a measure on $[-\pi, \pi]$, it satisfies

$$\int g(t + \pi) du_o(t) = - \int g(t) du_o(t)$$

for periodic functions g . That is, u_o is an odd measure. A straightforward adaptation of the "method of rotations" shows that $T_{m_o} \in B(L^p)$ for $1 < p < \infty$. See Theorem (2.6) and the proof preceding it in Chapter VI of [11], where the kernel is an odd function in $L^1(\mathbb{T})$. For the even multiplier m_e , the functions $\sin(\cdot)m_e$, $\cos(\cdot)m_e$ are both odd functions. We have

$$S * (\cos(\cdot)m_e) = (S * (\cos(\cdot)m))_o, \quad S * (\sin(\cdot)m_e) = (S * \sin(\cdot)m)_o,$$

and so $S * \sin(\cdot)m_e$ and $S * (\cos(\cdot)m_e)$ are in $BV(\mathbb{T})$. Hence $\sin(\cdot)m_e$ and $\cos(\cdot)m_e$ are also in $BV(\mathbb{T})$ and so are multipliers of kernels which are odd measures. As before, the corresponding operators are in $B(L^p)$; that is, the Riesz transforms

$$R_1 T_{m_e}, R_2 T_{m_e} \in B(L^p).$$

Each R_j is also in $B(L^p)$, and so also is each $R_j^2 T_{m_e}$. So $T_{m_e} \in B(L^p)$ as $R_1^2 + R_2^2 = -I$. As $T_m = T_{m_o} + T_{m_e}$, we have $T_m \in B(L^p)$ for $1 < p < \infty$. $\square$

In [9], Ricci and Weiss give a characterization of $H^1(\sum_{n=1}^\infty)$ that is particularly suitable to our presentation here. They consider Calderon-Zygmund singular operators (distributions) T with kernel $k(x) = u(x')/|x|^n$, where

$u \in L^1(\sum_{n-1})$ and has integral zero. By composing T with the Riesz transforms R_j ($1 \leq j \leq n$), one obtains n distributions that are each homogeneous of degree $-n$. In general, these distributions may or may not be given by kernels in $L^1(\sum_{n-1})$. Ricci and Weiss characterize $H^1(\sum_{n-1})$ by showing that $u \in H^1(\sum_{n-1})$ if and only if $R_j \circ T$ has a kernel $u_j \in L^1(\sum_{n-1})$. In the case of dimension 2, we know from Theorem 7 that the operators $R_i \circ U$ and $R_2 \circ U$ have kernels u_1 and u_2 in $L^1(\mathbb{T})$ if $U(=T_m)$ is in $B(H^1)$. Hence the following theorem results from the Ricci-Weiss result and our Theorem 7.

Theorem 10. *If U is a distribution that is homogeneous of degree -2 and has an extension in $B(H^1(\mathbb{R}^2))$, then the kernel u of U is in $H^1(\mathbb{T})$.*

Note that Corollary 8 follows immediately from Theorem 10.

It is well known that if $u \in H^1(\mathbb{T})$, then the Calderon-Zygmund singular integral T_m with kernel u is bounded on L^p for $1 < p < \infty$, and of course, $\sum_{n \neq 0} \frac{|\hat{u}(n)|}{|n|} < \infty$ holds. See for example Connett [3]. Thus $u \in H^1(\mathbb{T})$ is a sufficient condition for boundedness of T_m on L^p for $1 < p < \infty$. Theorem 10 gives a corresponding necessary condition, with H^1 replacing L^p .

Remarks on sufficient conditions for T_m to be in $B(H^1)$. The condition that m have an absolutely convergent Fourier series is not sufficient to guarantee that T_m extends to an operator on H^1 . This is easily seen by considering lacunary series and using Theorem 7. For example, if $m(\theta) = \sum e^{i\theta 2^n}/n^2$, then $|\hat{u}(2^n)| = 2^n/n^2$ and u cannot be a measure. In fact, by considering similar examples, one sees that conditions of the form

$$\sum n^\delta |\hat{m}(n)| < \infty$$

for $0 \leq \delta < 1$ are insufficient to guarantee that $T_m \in B(H^1)$. Conversely, Taibleson and Weiss [12] show that the condition

$$(6) \quad \sum |\hat{m}(n)|^2 n^4 < \infty$$

implies that T_m is bounded on H^p for $2/3 < p \leq 1$, by showing that T_m sends atoms to molecules. Daly [4] has shown that if T_m sends atoms to molecules boundedly, then condition (6) is satisfied. Taken together, the state of results on sufficient conditions for $T_m \in B(H^p)$ in terms of m appears to be these results by Daly, Taibleson, and Weiss and our Theorem 9.

The condition that $S * m$ is of bounded variation is not sufficient to imply that T_m is bounded on H^p ($0 < p \leq 1$). This can be seen by considering any odd function m that is of bounded variation, but not continuous, on $\mathbb{T}$.

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