LARGE GROUPS OF SYMMETRIES OF HANDLEBODIES

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ABSTRACT. Let V_g be an orientable three-dimensional handlebody with genus $g > 1$. Let $N(g)$ be the largest order among all finite groups which act effectively on V_g and preserve orientation. We show that $4(g+1) \leq N(g) \leq 12(g-1)$, and that $N(g)$ equals either $8(g-1)$ or $12(g-1)$ when g is odd. Moreover each of the indicated upper and lower bounds are achieved for infinitely many genera g . The techniques which are used lead to more detailed results and also specialize to yield similar results for compact surfaces with nonempty boundary.

0. INTRODUCTION

A well-known result of Hurwitz states that an orientation preserving symmetry group of a closed Riemann surface of genus $g > 1$ has order at most $84(g-1)$ (cf. §4.14 of [ZVC]). However for fixed values of g the precise determination of the maximal order of such a symmetry group is now recognized as an extremely intricate problem entailing an understanding of the indices of torsion-free normal subgroups of certain triangle groups. In a positive direction it was shown independently by Accola [A] and by Maclachlan [M₁] that this maximal order is bounded below by $8(g+1)$ and above by $84(g-1)$, and that each of these bounds is attained for infinitely many g 's. For some classes of symmetry groups more satisfactory results are possible. For instance Wiman (cf. [H]) proved that the order of the largest cyclic group acting on a genus g Riemann surface is $2(2g+1)$. Also the order of the largest abelian symmetry group is $4(g+1)$ [M₁], and the largest nilpotent symmetry group has order at most $16(g-1)$ [Z₃] (related results are derived in [Z₄]). In this note we will establish similar types of bounds for large groups of symmetries of handlebodies. In this connection it is a general principle that most of the largest group actions on a closed Riemann surface do not extend to a handlebody.

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Let V_g be an orientable three-dimensional handlebody of genus $g > 1$ and let G be a finite group. We consider G -actions on V_g , which are embeddings of G into the group $\text{Diff}^+(V_g)$ of orientation preserving diffeomorphisms of V_g . Let $N(g)$ be the largest order among all finite groups which act on V_g . In $[Z_1, Z_2, \text{MMZ}_1]$ it was shown that $N(g) \leq 12(g-1)$. We will prove in Theorem 1 that $4(g+1) \leq N(g) \leq 12(g-1)$ and that the lower and upper bounds are each attained for infinitely many g 's. In Theorem 2 we improve this result for odd values of g , showing that in this case $8(g-1) \leq N(g) \leq 12(g-1)$ where again the bounds are achieved for infinitely many g 's. Our approach will lead to an even more precise delineation of the problem of computing $N(g)$, as is indicated in §2 in the Remarks following Theorems 1 and 2. We will show that for each g there is an integer $n = n_g$ with $3 \leq n \leq g+1$ (but $n_g \leq 4$ when g is odd) so that $N(g) = [4n/(n-2)](g-1)$. Infinitely many values of n arise as n_g for some genus g but also infinitely many values of n do not arise in this way. Furthermore we will show that if n occurs as n_g for one g then it occurs as n_g for infinitely many g 's.

For special classes of groups there are more satisfactory results, as in the Riemann surface case. For instance it is shown in Theorem 7.3(c) of $[\text{MMZ}_1]$ that the order of the largest cyclic group acting on the handlebody V_g is $2g$, when g is odd, and $2(g+1)$, when g is even. From our proof of Theorem 1 in §2 it is evident that V_g always admits an action by the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_{g+1}$ of order $(2g+1)$, and an application of Theorem 7.6 of $[\text{MMZ}_1]$ shows that this is the largest possible symmetry of an abelian group on V_g except when $g = 5$; the handlebody V_5 admits the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as an abelian symmetry of largest possible order. Utilizing Proposition 2 of §1 it can be shown that the largest nilpotent group acting on V_g has order at most $8(g-1)$, and this upper bound is achieved whenever $g-1$ is a power of 2 (as the group constructed in Theorem 2 of §2 is nilpotent in this case). We also mention that $N(g) = 12(g-1)$ for $g = 2, 3$, or 4 (cf. Theorem 8.2 of $[\text{MMZ}_1]$).

In §3 we will show that our results on large symmetry groups of handlebodies specialize to yield all of the directly analogous results for compact surfaces with nonempty boundary with respect to the concept of algebraic genus. In particular we recover results of May $[\text{M}_2]$.

1. PRELIMINARIES

Consider a finite graph of finite groups $(\Gamma, \mathcal{G})$. If G is a finite group then a homomorphism $\rho: \pi_1(\Gamma, \mathcal{G}) \rightarrow G$ is said to be *finite-injective* provided that ρ is one-to-one on each finite subgroup; equivalently, ρ is injective on each vertex group of $\pi_1(\Gamma, \mathcal{G})$. In $[\text{MMZ}_1]$ a set of *normalized conditions* for graphs of groups is given which is used to study group actions on handlebodies. We say that $(\Gamma, \mathcal{G})$ is *G-admissible* if it satisfies the normalized conditions and there is a finite-injective epimorphism from $\pi_1(\Gamma, \mathcal{G})$ to G .

Proposition 1 (Theorem 6.1 [MMZ₁]). *A finite group G acts on V_g if and only if there is a G -admissible graph of groups $(\Gamma, \mathcal{G})$ with $g - 1 = -\chi(\Gamma, \mathcal{G})|G|$.*

Here $\chi(\Gamma, \mathcal{G})$ denotes the *Euler characteristic* of $(\Gamma, \mathcal{G})$ which is defined as

$$\sum \frac{1}{|G_v|} - \sum \frac{1}{|G_e|}$$

where the sums are taken over all vertex groups G_v and over all edge groups G_e for unoriented edges e . For example the graph of groups

$$\Gamma(H, C, K) = \begin{array}{c} C \\ \bullet \text{---} \bullet \\ H \quad K \end{array}$$

has $\chi(\Gamma(H, C, K)) = 1/|H| + 1/|K| - 1/|C|$ and $\pi_1(\Gamma, \mathcal{G}) = H *_C K$. In our setting we will only be interested in graphs of groups $(\Gamma, \mathcal{G})$ with $\chi(\Gamma, \mathcal{G}) \geq -\frac{1}{4}$; the relevant graphs are given in the next Proposition. We use the notation $\Gamma_n = \Gamma(\mathbf{D}_2, \mathbf{Z}_2, \mathbf{D}_n)$ where $\mathbf{D}_n = \mathbf{Z}_n \circ \mathbf{Z}_2$ is the dihedral group of order $2n$ (the *semidirect product* of $\mathbf{Z}_n$ with $\mathbf{Z}_2$) and the $\mathbf{Z}_2$ edge injects to an order two subgroup not contained in $\mathbf{Z}_n \circ 1$. (Any two such graphs of groups have isomorphic fundamental groups.)

Proposition 2 (Theorem 7.1 [MMZ₁]). *If $(\Gamma, \mathcal{G})$ is a graph of groups satisfying the normalized conditions and having $0 > \chi(\Gamma, \mathcal{G}) > -\frac{1}{4}$ then $(\Gamma, \mathcal{G})$ is either Γ_n for some $n \geq 3$ or one of the following 17 graphs*

$(\Gamma, \mathcal{G})$	$\chi(\Gamma, \mathcal{G})$	$(\Gamma, \mathcal{G})$	$\chi(\Gamma, \mathcal{G})$	$(\Gamma, \mathcal{G})$	$\chi(\Gamma, \mathcal{G})$
(1) $\Gamma(\mathbf{D}_3, \mathbf{Z}_3, \mathbf{A}_4)$	$-\frac{1}{12}$	(6) $\Gamma(\mathbf{Z}_2, 1, \mathbf{Z}_3)$	$-\frac{1}{6}$	(12) $\Gamma(\mathbf{D}_2, \mathbf{Z}_2, \mathbf{S}_4)$	$-\frac{5}{24}$
(2) $\Gamma(\mathbf{D}_4, \mathbf{Z}_4, \mathbf{S}_4)$	$-\frac{1}{12}$	(7) $\Gamma(\mathbf{D}_2, \mathbf{Z}_2, \mathbf{A}_4)$	$-\frac{1}{6}$	(13) $\Gamma(\mathbf{D}_3, \mathbf{Z}_2, \mathbf{D}_4)$	$-\frac{5}{24}$
(3) $\Gamma(\mathbf{D}_5, \mathbf{Z}_5, \mathbf{A}_5)$	$-\frac{1}{12}$	(8) $\Gamma(\mathbf{D}_3, \mathbf{Z}_2, \mathbf{D}_3)$	$-\frac{1}{6}$	(14) $\Gamma(\mathbf{A}_4, \mathbf{Z}_3, \mathbf{S}_4)$	$-\frac{5}{24}$
(4) $\Gamma(\mathbf{D}_3, \mathbf{Z}_3, \mathbf{S}_4)$	$-\frac{1}{8}$	(9) $\Gamma(\mathbf{A}_4, \mathbf{Z}_3, \mathbf{A}_4)$	$-\frac{1}{6}$	(15) $\Gamma(\mathbf{D}_2, \mathbf{Z}_2, \mathbf{A}_5)$	$-\frac{7}{30}$
(5) $\Gamma(\mathbf{D}_3, \mathbf{Z}_3, \mathbf{A}_5)$	$-\frac{3}{20}$	(10) $\Gamma(\mathbf{S}_4, \mathbf{Z}_4, \mathbf{S}_4)$	$-\frac{1}{6}$	(16) $\Gamma(\mathbf{D}_3, \mathbf{Z}_2, \mathbf{D}_5)$	$-\frac{7}{30}$
		(11) $\Gamma(\mathbf{A}_5, \mathbf{Z}_5, \mathbf{A}_5)$	$-\frac{1}{6}$	(17) $\Gamma(\mathbf{A}_4, \mathbf{Z}_3, \mathbf{A}_5)$	$-\frac{7}{30}$

2. LARGE SYMMETRIES ON HANDLEBODIES

We now have the necessary preliminaries to begin our description of $N(g)$. In the subsequent discussions we shall consider the set

$$\mathcal{F} = \{g \in \mathbb{Z}^+ | g > 12, g - 1 \text{ is a prime, and } g \not\equiv 2 \pmod{3}\}.$$

Theorem 1. *For each integer $g > 1$ we have $N(g) \geq 4(g + 1)$, and for the infinitely many values of g in $\mathcal{F}$, $N(g) = 4(g + 1)$.*

Proof. For any integer g let $G = \mathbf{Z}_2 \times \mathbf{D}_{g+1}$. There is a finite-injective epimorphism ρ from $\pi_1(\Gamma_{g+1}) = \mathbf{D}_2 *_2 \mathbf{D}_{g+1}$ to G with $\rho(\mathbf{D}_{g+1}) = 1 \times \mathbf{D}_{g+1}$ and $\rho(\mathbf{D}_2) = \mathbf{Z}_2 \times \mathbf{Z}_2$. By Proposition 1 it follows that G acts on the handlebody of genus $-\chi(\Gamma_{g+1})|G| + 1 = [(g - 1)/4(g + 1)]4(g + 1) + 1 = g$. We conclude that $N(g) \geq |G| = 4(g + 1)$.

To prove the second part of the Theorem, suppose that $g \in \mathcal{F}$ and that V_g admits a G -action for some finite groups G with $|G| > 4(g + 1)$. By Proposition 1 there is a G -admissible graph of groups $(\Gamma, \mathcal{G})$ with $\chi(\Gamma, \mathcal{G}) = -(g - 1)/|G|$.

As $-(g-1)/|G| > -(g-1)/4(g+1) > -\frac{1}{4}$, $(\Gamma, \mathcal{G})$ must be one of the graphs listed in Proposition 2. Let $\rho: \pi_1(\Gamma, \mathcal{G}) \rightarrow G$ be a finite-injective epimorphism. We first consider the graphs $(\Gamma, \mathcal{G}) = \Gamma_n$ which have $\pi_1(\Gamma, \mathcal{G}) = \mathbf{D}_2 *_{\mathbf{Z}_2} \mathbf{D}_n$. The Euler characteristic is $\chi(\Gamma, \mathcal{G}) = -(n-2)/4n$ and using this we write

$$g-1 = \frac{n-2}{4n}|G| = \frac{n-2}{d} \frac{d|G|}{4n}$$

where $d = 1$ if n is odd, and $d = 2$ if n is even. As $g-1$ is a prime integer and $2n$ divides $|G|$ (G contains a subgroup $\rho(\mathbf{D}_n)$ with order $2n$), both $d|G|/4n$ as well as $(n-2)/d$ are integers and one of them must equal 1. If $d|G|/4n = 1$ then $g-1 = (n-2)/d$ and

$$4(g+1) = 4\left(\frac{n-2}{d} + 2\right) = \frac{4n + 8(d-1)}{d} \geq \frac{4n}{d} = |G|$$

which contradicts the choice of G . Therefore we must have $(n-2)/d = 1$ implying that $n = 3$ or $n = 4$. We will investigate these possibilities in Cases A and C below, but next we consider the 17 graphs $(\Gamma, \mathcal{G}) \neq \Gamma_n$ which are listed in Proposition 2. Suppose $\chi(\Gamma, \mathcal{G}) = -A/B$ where A and B are relatively prime integers; then $A \leq 7$ and

$$g-1 = -\chi(\Gamma, \mathcal{G})|G| = \frac{A}{B}|G|.$$

If $A \neq 1$ then using the primality of $g-1$ we have $|G| = B$ and $g-1 = A \leq 7$. This contradicts the fact that $g > 12$. Thus it must be that $A = 1$. Referring to Proposition 2 we see that $|G| = B(g-1)$ where B is 6, 8 or 12, and the graphs (5) and (12)–(17) are ruled out. Since $g-1$ is a large prime, G cannot contain subgroups with order 60 or 24 (such as $\mathbf{A}_5$ or $\mathbf{S}_4$). Therefore $(\Gamma, \mathcal{G})$ cannot be any of the graphs (2), (3), (4), (10), or (11) from Proposition 2. Also the graphs (7) and (9) are ruled out since here $|G| = 6(g-1)$ and G cannot contain an order 12 subgroup (such as $\mathbf{A}_4$). This leaves the graphs (1), (6) and (8), as well as Γ_3 and Γ_4 which must be considered individually.

Case A. $(\Gamma, \mathcal{G}) = \Gamma_3$. Then $|G| = 12(g-1)$ and $\pi_1(\Gamma, \mathcal{G}) = \mathbf{D}_2 *_{\mathbf{Z}_2} \mathbf{D}_3$. A Sylow $(g-1)$ -subgroup P of G is cyclic with order $g-1$ and by Sylow's Theorem the number of such subgroups divides 12 and is congruent to 1 mod $(g-1)$. Since $g-1 \geq 12$ it follows that P is normal in G . The well-known Schur-Zassenhaus Theorem states that in a finite group any normal subgroup has a complement if its order is relatively prime to its index. In our setting this implies that P must have a complement, i.e. $-G$ is a semidirect product $P \circ \overline{G}$ where $\overline{G}$ has order 12. (Our usage of this Theorem can be avoided by elementary arguments but we include it here, and in the sequel, for the sake of expediency.) Under the projection from G to $\overline{G}$ the image of $\rho(\mathbf{D}_3)$ is a $\mathbf{D}_3$ subgroup and hence $\overline{G} \cong \mathbf{D}_6$ as $\mathbf{D}_6$ is the only group of order 12 which contains such a subgroup. Now $\text{Aut}(P)$ is cyclic (in fact $\cong \mathbf{Z}_{g-2}$) and the action of $\overline{G}$ on P must factor through $(\mathbf{D}_6)_{ab} \cong \mathbf{D}_2$. It follows that the unique $\mathbf{Z}_3$ subgroup H of $\overline{G}$ acts trivially on P and as a result H is the only order 3 subgroup of

G . (If $(xy)^3 = 1$ where $x \in P$ and $y \in G$ then $y^3 = 1$ implying that x^3 and then x are trivial.) The image of ρ is generated by $\rho(\mathbf{D}_2)$ together with H ; as H is normal this image has order 12 so that ρ is not surjective. From this contradiction we conclude that Case A cannot arise.

Case B. $(\Gamma, \mathcal{F}) = \Gamma(\mathbf{D}_3, \mathbf{Z}_3, \mathbf{A}_4)$. Then $|G| = 12(g-1)$ and $\pi_1(\Gamma, \mathcal{F}) = \mathbf{D}_3 *_{\mathbf{Z}_3} \mathbf{A}_4$. Let P be a Sylow $(g-1)$ -subgroup of G . As before the Sylow theory implies that P is normal in G . Since $\rho(\mathbf{A}_4)$ has order 12 it follows that $G = P \circ \rho(\mathbf{A}_4)$. The $\rho(\mathbf{A}_4)$ action on P factors through $(\mathbf{A}_4)_{ab} \cong \mathbf{Z}_3$ so every order 2 element of $\rho(\mathbf{A}_4)$ commutes with P . If $xy(x \in P, y \in \rho(\mathbf{A}_4))$ is an element of G with order two then $y^2 = 1$ and $1 = (xy)^2 = x^2$ so that $x = 1$. This shows that every order two element is contained in $\rho(\mathbf{A}_4)$. In particular it follows that $\rho(\mathbf{D}_3) \subset \rho(\mathbf{A}_4)$ which is impossible as $\mathbf{A}_4$ has no $\mathbf{D}_3$ subgroup.

Case C. $(\Gamma, \mathcal{F}) = \Gamma_4$. Then $|G| = 8(g-1)$ and $\pi_1(\Gamma, \mathcal{F}) = \mathbf{D}_2 *_{\mathbf{Z}_2} \mathbf{D}_4$. Again we have a cyclic normal Sylow subgroup P of G with order $g-1$. Since $|\rho(\mathbf{D}_4)| = 8$ we must have $G = P \circ_{\psi} \rho(\mathbf{D}_4)$ where $\psi: \rho(\mathbf{D}_4) \rightarrow \text{Aut}(P)$ is the semidirect product action. Choose elements $a \in \rho(\mathbf{D}_4)$ and $b \in \rho(\mathbf{D}_4) \cap \rho(\mathbf{D}_2)$ so that $a^4 = b^2 = 1$ and $bab^{-1} = a^{-1}$. Suppose that $\rho(\mathbf{D}_2)$ is the subgroup $\langle b, xy \rangle$ generated by b and xy where $x \in P$ and $y \in \rho(\mathbf{D}_4)$. Then $xy = b(xy)b^{-1} = \psi(b)(x)byb^{-1}$, which implies that $\psi(b)(x) = x$ and $byb^{-1} = y$. This shows that y is an element of the centralizer $\langle b, a^2 \rangle$ of b in $\rho(\mathbf{D}_4)$, and that if $x \neq 1$ then $b \in \ker(\psi)$. Since $a^2 \in \ker(\psi)$ it follows that either $y \in \ker(\psi)$ or $x = 1$. But xy has order two so $1 = (xy)^2 = x^2y^2$ and it must be that $x = 1$. Therefore $\rho(\mathbf{D}_2) = \langle b, y \rangle \subset \rho(\mathbf{D}_4)$ which contradicts the fact that ρ is surjective.

Case D. $(\Gamma, \mathcal{F}) = \Gamma(\mathbf{Z}_2, 1, \mathbf{Z}_3)$. Then $|G| = 6(g-1)$ and $\pi_1(\Gamma, \mathcal{F}) = \mathbf{Z}_2 * \mathbf{Z}_3$. Applying the Sylow theory as before, we see that G contains a normal cyclic subgroup P of order $g-1$. By the Schur-Zassenhaus Theorem, $G = P \circ \overline{G}$ where $\overline{G}$ has order 6. Since $\text{Aut}(P) \cong \mathbf{Z}_{g-2}$ and $g-2 \not\equiv 0 \pmod{3}$ each order three element of G acts trivially on P . Therefore $\rho(\mathbf{Z}_3)$ is normal in G and the image of ρ equals $\langle \rho(\mathbf{Z}_3), \rho(\mathbf{Z}_2) \rangle$ which has order 6. This contradicts the fact that ρ is surjective.

Case E. $(\Gamma, \mathcal{F}) = \Gamma(\mathbf{D}_3, \mathbf{Z}_2, \mathbf{D}_3)$. Then $|G| = 6(g-1)$ and $\pi_1(\Gamma, \mathcal{F}) = \mathbf{D}_3 *_{\mathbf{Z}_2} \mathbf{D}_3$. Again we have $G = P \circ \rho(\mathbf{D}_3)$ where P is cyclic of order $g-1$. Since $(\mathbf{D}_3)_{ab} \cong \mathbf{Z}_2$ each element of order three in $\rho(\mathbf{D}_3)$ acts trivially on P . It follows that each element of order three is contained in $\rho(\mathbf{D}_3)$ and that $\rho(\mathbf{D}_3)$ is the image of ρ , contradicting the fact that ρ is surjective.

We have now shown that if $g \in \mathcal{F}$ then V_g admits no G -action where $|G| > 4(g+1)$ and the proof of the Theorem is complete. $\square$

Remarks (1). The Euler characteristic of Γ_n is $-(n-2)/4n$ and a quick glance at the 17 cases listed in Proposition 2 shows that all graphs of groups $(\Gamma, \mathcal{F})$ with $0 > \chi(\Gamma, \mathcal{F}) > -\frac{1}{4}$ have $\chi(\Gamma, \mathcal{F}) = -(n-2)/4n$ for some $n \geq 3$. It

follows from Theorem 1 that for every $g > 1$ we have $N(g) = [4n/(n-2)(g-1)]$ for some $n = n_g$ with $3 \leq n \leq g+1$, and that $n_g = g+1$ when $g \in \mathcal{F}$.

(2) In the proof of Theorem 1, it was only in Case D that we used the property that $g \not\equiv 2 \pmod{3}$ when $g \in \mathcal{F}$. Consider the complementary set

$$\mathcal{F}' = \{g \in \mathbb{Z}^+ | g > 12, g-1 \text{ is a prime, and } g \equiv 2 \pmod{3}\}.$$

If $g \in \mathcal{F}'$ then the group $G = \mathbf{Z}_{g-1} \circ_{\psi} \mathbf{Z}_6$ whose action $\psi: \mathbf{Z}_6 \rightarrow \text{Aut}(\mathbf{Z}_{g-1})$ is one-to-one, is easily seen to be a finite-injective quotient of $\mathbf{Z}_2 * \mathbf{Z}_3$. (If $\mathbf{Z}_{g-1} \circ 1 = \langle a \rangle$ and $1 \circ \mathbf{Z}_6 = \langle t \rangle$ then G is generated by $\langle t^3 \rangle \cong \mathbf{Z}_2$ together with $\langle at^2 \rangle \cong \mathbf{Z}_3$.) This shows that

$$N(g) = 6(g-1) \text{ for the infinitely many values of } g \text{ in } \mathcal{F}'.$$

In other words, $n_g = 6$ when $g \in \mathcal{F}'$.

Theorem 2. *If g is an odd integer then $N(g) \geq 8(g-1)$. For the infinitely many integers of the form $g = 2p+1$ where $p > 23$ is prime we have $N(g) = 8(g-1)$.*

Proof. For each odd integer g we will construct a group

$$G = (\mathbf{Z}_{(g-1)/2} \circ_{\phi} \mathbf{D}_4) \circ_{\psi} \mathbf{Z}_2$$

which is a finite-injective quotient of $\pi_1(\Gamma_4) = \mathbf{D}_2 *_{\mathbf{Z}_2} \mathbf{D}_4$. By Proposition 1 and since $\chi(\Gamma_4) = -\frac{1}{8}$ it follows that G acts on the handlebody of genus $\frac{1}{8}|G| + 1 = g$ and $N(g) \geq |G| = 8(g-1)$. We now describe the actions ϕ and ψ used to define G . Writing $\mathbf{Z}_{(g-1)/2} = \langle x | x^{(g-1)/2} = 1 \rangle$, $\mathbf{D}_4 = \langle y, z | y^4 = z^2 = 1, zyz^{-1} = y^{-1} \rangle$, and $\mathbf{Z}_2 = \langle t | t^2 = 1 \rangle$, the actions are given by

$$\phi: \begin{cases} yxy^{-1} = x^{-1} \\ zxz^{-1} = x \end{cases} \quad \psi: \begin{cases} txt^{-1} = x^{-1} \\ tyt^{-1} = xy \\ tzt^{-1} = z \end{cases}$$

and it is straightforward to check that they are well defined. It remains to show that Γ_4 is G -admissible. Write $\mathbf{D}_2 *_{\mathbf{Z}_2} \mathbf{D}_4 = \langle T, Y, Z \rangle$ where $T^2 = Z^2 = Y^4 = 1$, $ZTZ^{-1} = T$, and $ZYZ^{-1} = Y^{-1}$. Then a homomorphism $\rho: \mathbf{D}_2 *_{\mathbf{Z}_2} \mathbf{D}_4 \rightarrow G$ is given by $\rho(T) = t$, $\rho(Y) = y$ and $\rho(Z) = z$. Clearly ρ is finite-injective and it is surjective since $tyt^{-1}y^{-1} = x$. This completes the proof of the first part of the Theorem.

Now suppose that $g = 2p+1$ where $p > 23$ is prime, and suppose that V_g admits a G -action with $|G| > 8(g-1)$. Then there is a G -admissible graph of groups $(\Gamma, \mathcal{G})$ with $\chi(\Gamma, \mathcal{G}) = -(g-1)/|G| > -\frac{1}{8}$. Therefore $\chi(\Gamma, \mathcal{G}) = -\frac{1}{12}$, $(\Gamma, \mathcal{G})$ is either Γ_3 or one of the first 2 of the 17 graphs listed in Proposition 2, and $|G| = 12(g-1) = 24p$. As before (since $p > 23$) a Sylow p -subgroup P of G is normal and $G = P \circ_{\phi} \overline{G}$ where $|\overline{G}| = 24$. The composition of a finite-injective epimorphism $\rho: \pi_1(\Gamma, \mathcal{G}) \rightarrow G$ with the projection $G \rightarrow \overline{G}$ is again a finite-injective epimorphism since no element of $\pi_1(\Gamma, \mathcal{G})$ has order

$p > 23$. By Proposition 1, $\overline{G}$ acts on the handlebody of genus $\frac{1}{12}|\overline{G}| + 1 = 3$. But by Theorem 8.2(c) of [MMZ₁] the only group of order 24 acting on V_3 is S_4 , so $\overline{G} \cong S_4$. The action $\phi: \overline{G} \rightarrow \text{Aut}(P)$ factors through $(S_4)_{ab} \cong Z_2$. If ϕ is trivial then $G = P \times \overline{G}$ and it readily follows that $\text{image}(\rho) \subseteq 1 \times \overline{G}$ contradicting the fact that ρ is surjective. Thus we assume that ϕ is nontrivial which implies that $\ker(\phi)$ has index two in $\overline{G}$ and $\ker(\phi) \cong A_4$. We now consider the three possible graphs $(\Gamma, \mathcal{G})$ individually.

Case A. $(\Gamma, \mathcal{G}) = (\Gamma(D_4, Z_4, S_4))$ and $\pi_1(\Gamma, \mathcal{G}) = D_4 *_{Z_4} S_4$. We may assume that $\rho(S_4) = 1 \circ \overline{G} \subseteq P \circ \overline{G} = G$. Suppose that $\rho(D_4)$ is generated by $\langle z, xw \rangle$ where $z, w \in \overline{G}$, $x \in P$, z has order 4, xw has order 2 and $(xw)z(xw)^{-1} = z^{-1}$. But then $z^{-1} = x(wzw^{-1})x^{-1} = x^2(wzw^{-1})$ since $wzw^{-1} \notin \ker(\phi)$ (A_4 contains no element of order 4). From this it follows that $x^2 = 1$ and $x = 1$. Since $\text{image}(\rho) = \langle xw, \overline{G} \rangle \subseteq \overline{G}$ this contradicts the hypothesis that ρ is surjective.

Case B. $(\Gamma, \mathcal{G}) = (\Gamma(D_3, Z_3, A_4))$ and $\pi_1(\Gamma, \mathcal{G}) = D_3 *_{Z_3} A_4$. We first observe that $\rho(A_4) \subseteq 1 \circ \overline{G}$ since if xw has order 2 or 3 where $x \in P$ and $w \in \ker(\phi)$ then $x = 1$ (since either $x^2 = 1$ or $x^3 = 1$). Now suppose that xw ($x \in P$, $w \in \overline{G}$) is an element of order 2 in $\rho(D_3 - Z_3)$. Since ρ is surjective we must have $w \notin \ker(\phi)$. Let $y \in \ker(\phi)$ be an element of order 3 so that wy has order 4 (thus $\overline{G}$ is generated by $\{w, y\}$). Then $(xwy)^2 = x(wy)x(wy) = (wy)^2$ so xwy has order 4. This implies that $\{xw, y\}$ generates an S_4 subgroup of G and this subgroup contains $\rho(A_4)$ (which is generated by $\{(wy)^2, y\}$). Hence the image of ρ , which is generated by $\rho(A_4)$ together with xw , equals this S_4 subgroup and ρ is not surjective.

Case C. $(\Gamma, \mathcal{G}) = \Gamma_3$ and $\pi_1(\Gamma, \mathcal{G}) = D_2 *_{Z_2} D_3$. Let W , Y , and Z be elements of $D_2 *_{Z_2} D_3$ so that $D_2 = \langle W, Y | W^2 = Y^2 = (WY)^2 = 1 \rangle$ and $D_3 = \langle Y, Z | Y^2 = Z^3 = 1, YZY^{-1} = Z^{-1} \rangle$. Let $\rho(W) = x_1w$, $\rho(Y) = x_2y$ and $\rho(Z) = x_3z$ where $x_1, x_2, x_3 \in P$ and $w, y, z \in \overline{G}$. Note that $z \in \ker(\phi)$ (A_4 contains all order 3 element of S_4), $y \notin \ker(\phi)$ (otherwise A_4 would contain a D_3 subgroup) and $w \in \ker(\phi)$ (if not then replace W with WY). As shown in Case B, we must have $x_1 = x_3 = 1$ so $\langle \rho(W), \rho(Z) \rangle = 1 \circ \ker(\phi) = A_4$. But $\langle \rho(W), \rho(Z) \rangle$ has index 2 in $\text{image}(\rho)$, so ρ is not surjective. $\square$

Remarks (1). Theorem 2 implies that if g is odd then either $N(g) = 8(g-1)$ or $N(g) = 12(g-1)$, and that $N(g) = 8(g-1)$ for infinitely many odd integers. We now show that also $N(g) = 12(g-1)$ for infinitely many odd integers. This was shown in [Z₂] but we will derive some further results from our argument. This type of argument is also employed in [GM].

Let $g > 1$ be an integer and suppose that $N(g) = A(g-1)$. (Thus $A = 4n/(n-2)$ where $n = n_g$.) Then there is a finite group G with $|G| = N(g)$ and a G -admissible graph of groups $(\Gamma, \mathcal{G})$ such that $\chi(\Gamma, \mathcal{G}) = -1/A$. This

determines a group extension

$$1 \rightarrow F_g \rightarrow \pi_1(\Gamma, \mathcal{G}) \rightarrow G \rightarrow 1$$

where F_g is free of rank g . For any positive integer m let K_m be the kernel of the projection $F_g \rightarrow H_1(F_g; \mathbb{Z}_m)$. Then K_m is characteristic in F_g and normal in $\pi_1(\Gamma, \mathcal{G})$. Let $G_m = \pi_1(\Gamma, \mathcal{G})/K_m$; this is an extension of G by $\mathbb{Z}_m^g$ and has order $|G|m^g = A(g-1)m^g$. Since the projection $\pi_1(\Gamma, \mathcal{G}) \rightarrow G_m$ is finite-injective, there is an action of G_m on the handlebody of genus $g_m = -\chi(\Gamma, \mathcal{G})|G_m| + 1 = (1/A)|G|m^g + 1 = (g-1)m^g + 1$. We conclude that, for each positive integer m :

If $N(g) = A(g-1)$ and $g_m = (g-1)m^g + 1$ then $N(g_m) \geq A(g_m - 1)$.

In particular, if $N(g) = 12(g-1)$ then $N(g_m) = 12(g_m - 1)$ for each m . To illustrate, since $N(g) = 12(g-1)$ for $g = 2, 3$, or 4 it follows that $N(g') = 12(g' - 1)$ whenever $g' = m^2 + 1$, $2m^3 + 1$, or $3m^4 + 1$ for any positive integer m .

(2) Expanding on the previous remark, we show that:

If $n = n_g$ for some g then $n = n_g$ for infinitely many g 's.

Suppose that $n = n_g$ for some genus g and let $A = 4n/(n-2)$, so that $N(g) = A(g-1)$. Then $n \geq 3$ and $-1/A > -\frac{1}{4}$. By Proposition 2 there are only finitely many graphs of groups $(\Gamma, \mathcal{G})$ satisfying the normalized conditions and having $\chi(\Gamma, \mathcal{G}) \geq -1/A$. Let $p > 12(g-1)$ be any prime which does not divide the order of a vertex group of any of these graphs. (Thus p does not divide the numerator or denominator of the reduced rational number form of the Euler characteristic of any of these graphs.) Let $g_p = (g-1)p^g + 1$. By (1) of Remarks, $N(g_p) \geq A(g_p - 1)$. Suppose that $N(g_p) > A(g_p - 1)$, then there is a finite group G together with graph of groups $(\Gamma, \mathcal{G})$ and a finite-injective epimorphism $\rho: \pi_1(\Gamma, \mathcal{G}) \rightarrow G$ for which $|G| = -(g_p - 1)/\chi(\Gamma, \mathcal{G}) = -(g-1)p^g/\chi(\Gamma, \mathcal{G})$ and $-1/\chi(\Gamma, \mathcal{G}) > A$. Let P be a Sylow p -subgroup of G . As p divides neither the numerator nor the denominator of $-(g-1)/\chi(\Gamma, \mathcal{G})$, P has order p^g . Using Sylow's Theorem and the inequalities $p > 12(g-1) > -(g-1)/\chi(\Gamma, \mathcal{G})$, P is normal. The composition of ρ with the projection $G \rightarrow G/P$ gives an epimorphism $\pi_1(\Gamma, \mathcal{G}) \rightarrow G/P$ which is finite-injective because no finite subgroup of $\pi_1(\Gamma, \mathcal{G})$ has order p . It follows that the factor group G/P acts on the handlebody of genus $-\chi(\Gamma, \mathcal{G})|G/P| + 1 = g$. Also $A(g-1) = N(g) \geq |G/P| = -(g-1)/\chi(\Gamma, \mathcal{G})$ from which we obtain the contradiction $A \geq -1/\chi(\Gamma, \mathcal{G})$.

(3) We have now seen that for each g the rational number $N(g)/(g-1)$ equals $4n/(n-2)$ for some $n \geq 3$, and that if $4n/(n-2)$ occurs once then it occurs for infinitely many genera g . On the other hand:

Infinitely many values of $4n/(n-2)$ do not occur as $N(g)/(g-1)$ for any g .

As an example, if $n \equiv 2 \pmod{16}$ and $n \neq 2$ the values $4n/(n-2)$ do not arise in this way. To demonstrate this, suppose that $N(g) = 4n(g-1)/(n-2)$ for some g where n satisfies these conditions. Then $(n-2)N(g) = 4n(g-1)$. Since $(n-2)N(g) \equiv 0 \pmod{16}$ and $4n(g-1) \equiv 8(g-1) \pmod{16}$ we must have $8(g-1) \equiv 0 \pmod{16}$ or, equivalently, $g-1$ is even. Then Theorem 2 implies that $4n/(n-2) = N(g)/(g-1)$ is at least 8 which contradicts the hypothesis that $n > 2$.

3. LARGE SYMMETRIES ON BOUNDED SURFACES

We conclude by showing that the arguments of §2 readily lead to analogous results for actions on bounded surfaces. A *bounded surface* is a compact 2-manifold with nonempty boundary (either orientable or not) and its *algebraic genus* is the rank of its fundamental group. Let $N_2(g)$ be the maximal order of all finite groups acting on bounded surfaces of algebraic genus g (possibly reversing orientation). To study actions on bounded surfaces it is convenient to use an approach similar to that given in Proposition 1 for actions on handlebodies. There is a set of conditions on $(\Gamma, \mathcal{S})$, which we refer to as the *normalized conditions for surface actions*, so that a group G acts on some bounded surface with algebraic genus g if and only if G is the finite-injective quotient of $\pi_1(\Gamma, \mathcal{S})$ for such a graph. (This result is proved in [MMZ₂].) The normalized conditions for surface actions imply the normalized conditions (for handlebodies), so that relevant graphs with $0 > \chi(\Gamma, \mathcal{S}) > -\frac{1}{4}$ may be found in Proposition 2. Explicitly, these graphs are all of the Γ_n graphs, as well as graphs (6), (8), (13) and (16).

Corollary. (a) $N_2(g) \geq 4(g+1)$ and $N_2(g) = 4(g+1)$ for infinitely many integers g .

(b) If g is odd then $N_2(g) \geq 8(g-1)$ and $N_2(g) = 8(g-1)$ for infinitely many odd integers.

Proof. In proving the two analogous inequalities for handlebody actions (Theorems 1 and 2) we used constructions involving the graphs Γ_{g+1} and Γ_4 . Since these satisfy the normalized conditions for surfaces, the inequalities in (a) and (b) follow. To complete the proof one observes that $N_2(g) \leq N(g)$: if G acts on a bounded surface S with algebraic genus g then G acts on V_g by taking a (possibly twisted to make the handlebody orientable) product of $\hat{S}$ with $[0, 1]$ and extending the group action in the natural way. $\square$

The procedure used in proving the Corollary can also be used to show that each of the properties derived in the remarks following Theorems 1 and 2 also holds for actions on bounded surfaces. In fact for all of the genera g we have considered here $N(g) = N_2(g)$. It remains open whether or not this equality always holds, although we suspect that it does not.

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